JWPSR Research Program
3 Pages on Homogenous Isotropic Turbulence

This is a 3-page outline of our analysis of ***Homogeneous Isotropic Turbulence*** (**HIT**) using the Wiener Machinery – Integral Polynomial Functionals based on the Wiener process. All random turbulent velocities and pressures can be ***exactly*** expressed with these functionals, which have deterministic kernels. The equations of motion then become deterministic equations in real functions of real variables. In other words, the ***randomness*** of turbulence is transformed out of the equations! See "JWPSR3PagesOnWienerMachinery.doc" for further details.

HIT is the ***Grand Exemplar*** of turbulence. It is ***homogeneous*** and ***ergodic*** on three space dimensions and decaying in time[1]. This flow has little practical significance, but great theoretical importance because of the work of Batchelor (et al.) on the statistical theory and Kolmogorov (et al.) on the energy cascade. There are also some very important computer results using **DNS** methods. ***Homogeneous*** means that the statistics are independent of position in 3-space; ***Ergodic*** means that space averages are equal to ensemble averages; ***Isotropic*** means the statistics are independent of rotation of the coordinate system. Many authors also take the statistics to be independent of ***reflections*** of the coordinate system, but neither the physics nor the mathematics requires this – nor do we.

The standard equations of motion for an incompressible Newtonian fluid (in physical units) are:

$$\begin{aligned} &\tfrac{\partial}{\partial t} u^i + u^k \tfrac{\partial}{\partial x^k} u^i - \nu \nabla^2 u^i + \tfrac{\partial}{\partial x^i} p = 0 \\ &\tfrac{\partial}{\partial x^k} u^k = 0 \end{aligned} \tag{1}$$

Viscosity "**ν**" is the only constitutive parameter, so these physical equations are no more complicated than the normalized equations. The advantage is that we do not have *a priori* to declare scales of **L**, **T**, and **V**.

<u>**Invariant Correlation Tensors:**</u> By Robertson's Theory of Invariants, the 2-point 2-velocity (**2P2V**) correlation tensor and its (proper) Fourier transform must have the form[2]:

$$\begin{aligned} R^{ij}(\xi^k,t) &\triangleq \mathcal{E}\left\{u^i(x^k)u^j(x^k+\xi^k)\right\} \\ &= R_p(\bar{\xi},t)\,\xi^i\xi^j + R_q(\bar{\xi},t)\,\delta^{ij} + R_r(\bar{\xi},t)\,\varepsilon^{ijk}\xi^k \\ \Phi^{ij}(\kappa^k,t) &= \Phi_p(\bar{\kappa},t)\,\kappa^i\kappa^j + \Phi_q(\bar{\kappa},t)\,\delta^{ij} + \Phi_r(\bar{\kappa},t)\,\varepsilon^{ijk}\kappa^k \end{aligned} \tag{2}$$

The $\mathbf{R_k(..)}$ and $\mathbf{\Phi_k(..)}$ are ***real*** functions of a ***positive real*** parameter. For invariance to reflection, set $\mathbf{R_r(..)=0}$. For an incompressible fluid, the divergence of the velocity must be zero, which requires

$$\Phi^{ij}(\kappa^k,t) = \Pi^{ij}(\kappa^k)\cdot\Phi_a(\bar{\kappa},t) + \varepsilon^{ijk}\tfrac{\kappa^k}{\bar{\kappa}}\cdot\Phi_b(\bar{\kappa},t) \qquad \text{where: } \Pi^{ij}(\kappa^k) = \left(\delta^{ij} - \tfrac{\kappa^i\kappa^j}{\kappa^p\kappa^p}\right) \tag{3}$$

A similar analysis of the 2-point 3-velocity (**2P3V**) correlation tensor and its transform yields:

$$\begin{aligned} S^{ijl}(\xi^k) &= \mathcal{F}_1\left[\Sigma^{ijl}(\kappa^k)\right] = \mathcal{E}\left\{u^i(x^k,t;\alpha)\cdot u^j(x^k,t;\alpha)\cdot u^l(x^k+\xi^k,t;\alpha)\right\} \\ \Sigma^{ijl}(\kappa^k,t) &= \left(\kappa^i\kappa^j\kappa^l - \left(\kappa^i\delta^{jl}+\kappa^j\delta^{il}\right)\tfrac{\bar{\kappa}^2}{2}\right)\cdot\Sigma_a(\bar{\kappa},t) + \left(\kappa^i\varepsilon^{jlk}\kappa^k + \kappa^j\varepsilon^{ilk}\kappa^k\right)\cdot\Sigma_b(\bar{\kappa},t) \end{aligned} \tag{4}$$

<u>**Analysis by HPF:**</u> Our first analysis uses Wiener's Homogeneous Polynomial Functionals (**HPF**'s) with a Wiener process defined on 3-space. The appropriate Wiener Integral form is:

$$\begin{aligned} f_p(x,y,z,t,\alpha) &\triangleq \int_S F_p(x+x_1, y+y_1, z+z_1, t)\cdot dr(x_1,y_1,z_1,\alpha) \\ &\triangleq \int_S F_p(x^k+x_1^k,t)\cdot dr(x_1^k,\alpha) \end{aligned} \tag{5}$$

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[1] We shall say "stationary" as synonymous with "homogeneous" in this context.

[2] This notation is traditional even though it does lead to some confusion, especially with "R".

Here the Wiener Process is defined in 3-space and the integral is taken over all space. We then use a vector Wiener process $\mathbf{r}^{\beta}(\mathbf{x}^{k},\boldsymbol{\alpha})$ with a span of 3 for the **HPF** expansion of the velocities to produce:

$$u^{i}(t,\alpha)=\begin{cases}+U_{0}=0\\+\int_{S}U_{1}^{i\beta}(x^{k}+x_{1}^{k},t)\cdot dr^{\beta}(x_{1}^{k},\alpha)\\+\iint_{S}U_{2}^{i\beta\gamma}(x^{k}+x_{1}^{k},x^{k}+x_{2}^{k},t)\cdot dr^{\beta}(x_{1}^{k},\alpha)\cdot dr^{\gamma}(x_{1}^{k},\alpha)\\+\dots\end{cases}\qquad(6)$$

This expansion is focused on the final stages of decay where the dissipation term dominates the inertial term. We have no *a priori* proof that this expansion even converges, but we will have *a posteriori* credible results.

Most important: the velocity kernels are square integrable over all space, so they have proper Fourier Transforms.

Application of the Equations of Motion: The equations of motion yield a set of coupled equations in the velocity and pressure kernels that can be solved in sequence. The first two equations – in spectral form – are:

$$\begin{array}{l|l}\partial_{t}U_{1}^{i\beta}(\kappa_{1}^{k},t)-\nu\left(\kappa_{1}^{p}\kappa_{1}^{p}\right)U_{1}^{i\beta}(\kappa_{1}^{k},t)+\left(j\kappa_{1}^{i}\right)P_{1}^{\beta}(\kappa_{1}^{k},t)=0 & \left(j\kappa_{1}^{p}\right)U_{1}^{p\beta}(\kappa_{1}^{k},t)=0\\ \hline \partial_{t}U_{2}^{i\beta\gamma}(\kappa_{1}^{k},\kappa_{2}^{k},t)-\nu\left(\kappa_{1+2}^{p}\kappa_{1+2}^{p}\right)U_{2}^{i\beta\gamma}(\kappa_{1}^{k},\kappa_{2}^{k},t) & \\ \quad+\left(j\kappa_{1+2}^{k}\right)P_{2}^{\beta\gamma}(\kappa_{1}^{k},\kappa_{2}^{k},t)=-\left(j\kappa_{1+2}^{q}\right)U_{1}^{q\beta}(\kappa_{1}^{k},t)U_{1}^{i\gamma}(\kappa_{2}^{k},t) & \left(j\kappa_{1+2}^{p}\right)U_{2}^{i\beta\gamma}(\kappa_{1}^{k},\kappa_{2}^{k},t)=0\end{array}\qquad(7)$$

$$\text{where:}\quad \kappa_{1+2}^{p}\triangleq\kappa_{1}^{p}+\kappa_{2}^{p}$$

Solution to the 1st Order Equation: The 1st order solution is:

$$\begin{aligned}&P_{1}^{\beta}(\kappa_{1}^{k},t)\equiv 0\\&U_{1}^{i\beta}(\kappa_{1}^{k},t)=A_{1}^{i\beta}(\kappa_{1}^{k})\cdot e^{-\nu\bar{\kappa}_{1}^{2}t}\quad\text{with: }\kappa_{1}^{j}\cdot A_{1}^{j\beta}(\kappa_{1}^{k})=0\quad\text{and:}\quad\bar{\kappa}_{1}^{2}=\kappa_{1}^{p}\kappa_{1}^{p}\end{aligned}\qquad(8)$$

The first order pressure is identically zero and the velocity *spectrum* has exponential time decay.

For the present, we presume that the 1st and 2nd (and higher) order solutions contribute independently to the **2P2V** correlation tensor. Then:

$$\Phi_{2}^{ij}(\kappa_{1}^{k},t)=A_{1}^{i\beta}(+\kappa_{1}^{k})\cdot e^{-\nu\bar{\kappa}_{1}^{2}t}A_{1}^{j\beta}(-\kappa_{1}^{k})\cdot e^{-\nu\bar{\kappa}_{1}^{2}t}=\Pi^{ij}(\kappa_{1}^{k})\cdot\Phi_{a}(\bar{\kappa}_{1},t)+\varepsilon^{ijk}\tfrac{\kappa_{1}^{k}}{\bar{\kappa}_{1}}\cdot\Phi_{b}(\bar{\kappa}_{1},t)\qquad(9)$$

A very general form of the **U**'s (including the alternator term) is:

$$U_{1}^{i\beta_{1}}(\kappa_{1}^{k},t)=\left(\Pi^{i\beta_{1}}(\kappa_{1}^{k})\cdot U_{\pi1}(\bar{\kappa}_{1}^{2})+\varepsilon^{i\beta_{1}p}\tfrac{j\kappa_{1}^{p}}{\bar{\kappa}_{1}}U_{\varepsilon1}(\bar{\kappa}_{1})\right)\cdot e^{-\nu\bar{\kappa}_{1}^{2}t}\qquad(10)$$

Then the resulting first term contribution to the 2P2V correlation tensor is:

$$\begin{aligned}\Phi_{2}^{ij}(\kappa_{1}^{k},t)&=U_{1}^{i\beta_{1}}(-\kappa_{1}^{k},t)\cdot U_{1}^{j\beta_{1}}(+\kappa_{1}^{k},t)\\&=A_{1}^{i\beta}(+\kappa_{1}^{k})\cdot e^{-\nu\bar{\kappa}_{1}^{2}t}A_{1}^{j\beta}(-\kappa_{1}^{k})\cdot e^{-\nu\bar{\kappa}_{1}^{2}t}\\&=\Pi^{ij}(\kappa_{1}^{k})\cdot\Phi_{a}(\bar{\kappa}_{1},t)+\varepsilon^{ijk}\kappa_{1}^{k}\cdot\Phi_{b}(\bar{\kappa}_{1},t)\\&=\begin{Bmatrix}+\Pi^{ij}(\kappa_{1}^{k})\cdot\left(U_{\pi1}(\bar{\kappa}_{1})\cdot U_{\pi1}(\bar{\kappa}_{1})+U_{\varepsilon1}(\bar{\kappa}_{1})\cdot U_{\varepsilon1}(\bar{\kappa}_{1})\right)\\-2\varepsilon^{ijp}\tfrac{j\kappa_{1}^{p}}{\bar{\kappa}_{1}}\cdot\left(U_{\pi1}(\bar{\kappa}_{1})\cdot U_{\varepsilon1}(\bar{\kappa}_{1})\right)\end{Bmatrix}\cdot e^{-2\nu\bar{\kappa}_{1}^{2}t}\end{aligned}\qquad(11)$$

The ***Energy Density Function*** is[3]:

$$\begin{aligned}&E_{T}(t)\triangleq\mathcal{E}\left\{\tfrac{1}{2}\cdot u^{p}(x^{k},t)\cdot u^{p}(x^{k}+0,t)\right\}=R^{pp}(0,t)=\left(\tfrac{1}{2\pi}\right)^{3}\int_{0}^{\infty}E_{D}(\kappa,t)d\kappa\\&E_{D}(\kappa,t)=(4\pi)\cdot\kappa^{2}\cdot\Phi_{a}(\kappa)\cdot e^{-2\nu\kappa^{2}t}\end{aligned}\qquad(12)$$

[3] These results parallel Batchelor, q.v. and are derived by using Isotropy and spherical coordinates.

The final stage of energy decay then depends on the structure of $\mathbf{\Phi_a(\kappa)}$ around the origin. Batchelor expands $\mathbf{\Phi^{ij}(\kappa^k)}$ as an ***analytic*** function of $\boldsymbol{\kappa}^i$ around the origin and concludes that $\mathbf{E_D \sim t^{-5/2}}$. Saffman expands $\mathbf{\Phi_a(\kappa)}$ in a power series in $\boldsymbol{\kappa}$ around the origin and concludes that $\mathbf{E_D \sim t^{-3/2}}$. Our results confirm both conclusions consistent with the assumptions made.

However, $\mathbf{\Phi_a(\kappa)}$ is a ***real*** function of a ***positive real*** variable. Nothing in the physics or math requires $\mathbf{\Phi_a(\kappa)}$ to be ***regular***, let alone ***analytic*** around the origin. Some examples follow:

$$\begin{array}{llll} \text{Batchelor} & \Phi_a \sim \kappa^2 & E_T \sim t^{-5/2} & \\ \text{Saffman} & \Phi_a \sim \kappa^0 & E_T \sim t^{-3/2} & \\ \text{Irregular} & \Phi_a \sim \exp(-1/\kappa) & E_T \sim \exp(-c \cdot t^{1/3}) & \text{bizarre exponential} \\ \text{Pseudo}-\text{K41} & \Phi_a \sim \left((\kappa_D)^{5/3} + (\kappa)^{5/3}\right)^{-1} & E_T \sim t^{-3/2} & \kappa^{-5/3}\ \text{Spectrum} \end{array} \quad (13)$$

Many other spectrum forms and decay laws are possible, entirely dependent on initial conditions. One should be very guarded about drawing conclusions from the energy spectrum alone without detailed knowledge of the initial conditions.

<u>Solution to the 2nd Order Equation:</u> There is both a ***Homogeneous Solution*** (no drives) and a ***Particular Solution*** (driven by the 1st order terms). The Homogeneous Solution leads to another set of **HPF** functionals of even order only – no Gaussian term! This is interesting, but a separate topic.

The 2nd order particular solution has $\mathbf{U_2(..)}$ kernels entirely determined by the $\mathbf{U_1(..)}$ kernels. Moreover, the $\mathbf{P_2(..)}$ pressure term is ***not*** zero! Finally, the $\mathbf{U_2(..)}$ contribution to the **2P2V** correlation tensor $\mathbf{\Phi_4(..)}$ is completely determined by $\mathbf{\Phi_2(..)}$ and it will always decay faster than the $\mathbf{\Phi_2(..)}$ term in the final stages.

Overall, the 2nd and higher **HPF** terms do not seem to contribute much to this analysis of the final stages of decay. However, it may well be that the **OPF** analysis will better represent the inertial range, in which case higher order terms will be quite important.

<u>Further Results:</u> There are many other results to present, e.g. the effect of the alternator terms, and the structure of the **2P3V** correlation tensor. One interesting result comes from the **MFD** case. Suppose the fluid is a good electrical conductor. Suppose that an Isotropic field is well established, and then at time $\mathbf{t_0}$, a substantial Magnetic field is turned on. The velocity field then becomes axisymmetric with the parallel velocities about double the perpendicular ones. Suppose further that at time $\mathbf{t_1}$, the Magnetic field is turned off. Question: does the velocity field return to Isotropy?

The answer is ***NO***! This is strong evidence that the "natural" final decay of turbulence is not necessarily isotropic. This casts doubt on this fundamental premise of many spectral arguments.

<u>Conclusions:</u> This is an analysis of ***Isotropic Turbulence*** with ***Wiener's Homogeneous Polynomial Functionals*** (**HPF**'s). We develop all the known theoretical results, and demonstrate many other solutions with differing decay and separation laws. We can even visualize the flow for a given instance of $\boldsymbol{\alpha}$ – the sampling parameter.

This is a first application of the Wiener Machinery to **HIT**. We expect many more results with the application of **OPF**'s and/or ***Gaussian Transforms***.

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(1043 words)