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JWPSR Research Program
3 Pages on the Wiener Machinery

This is a 3-page outline of the Wiener Machinery – Integral Polynomial Functionals based on the Wiener process, used in our studies of turbulence. All random turbulent velocities and pressures can be ***exactly*** expressed with these functionals, which have deterministic kernels. The equations of motion then become deterministic pde's in real functions of real variables. In other words, the ***randomness*** of turbulence is transformed out of the equations!

For us, ***fluid turbulence*** is the presence of ***random*** velocities, pressures, etc. At the most fundamental level (per Wiener) "A Random Physical Quantity (say a turbulent velocity) is a real function of space, time, and a real (sampling) parameter $\boldsymbol{\alpha \in [0,1]}$." Therefore, a velocity component is just $\mathbf{u(x,y,z,t,\alpha)}$, whose average is:

$$\mathcal{E}\left\{u(x,y,z,t,\alpha)\right\} = \bar{u}(x,y,z,t) = \int_0^1 u(x,y,z,t,\alpha)\cdot d\alpha \qquad (1)$$

So the parameter $\boldsymbol{\alpha}$ selects a particular path from the sample space, and averages are simple integrals on $\boldsymbol{\alpha}$ over the interval **[0,1]**.

That's about it!

It is not trivial to prove that such a construction exists, but in fact, this can be done for smooth random quantities – including all physical quantities in continuum physics. This is the formalism chosen by Wiener[1] and later used by Yaglom[2] and Lumley[3]. It is a simple representation of random quantities with all the helpful machinery of real analysis, especially Hilbert Spaces. We do not have to deal with measure theory, sigma algebras, etc.

Next, the ***Wiener Process*** $\mathbf{r(t,\alpha)}$ is just Brownian Motion centered at the origin. From this Wiener constructed:

$$f_p(t,\alpha) \triangleq \int_{-\infty}^{+\infty} F_p(t+s)\, dr(s,\alpha) \qquad (2)$$

This is the celebrated ***"Wiener Stochastic Convolution Integral"***. It is an ordinary Riemann-Stieltjes integral[4], which exists because $\mathbf{r(t,\alpha)}$ is continuous and $\mathbf{F_p(..)}$ is locally bounded in variation. Note particularly that $\mathbf{f_p(t,\alpha)}$ is ***Gaussian***, ***stationary*** and ***ergodic*** on **t**. Moreover, the correlation function is:

$$\Phi_{pq}(\tau) \triangleq \mathcal{E}\left[f_p(t,\alpha)\cdot f_q(t+\tau,\alpha)\right]$$

$$\triangleq \begin{cases} \triangleq \int_0^1 \left(f_p(t,\alpha)\cdot f_q(t+\tau,\alpha)\right) d\alpha & \text{The "Ensemble" Average} \\ \triangleq \lim\limits_{A,B\to\infty} \dfrac{1}{A+B}\int_{-A}^{+B} \left(f_p(s)\cdot f_q(s+\tau)\right) ds & \text{The "Time" Average} \end{cases} \qquad (3)$$

So the $\mathbf{F_p(..)}$ are $\mathbf{L^2}$ and have ***proper*** Fourier Transforms. Finally, the $\mathbf{f_p(t,\alpha)}$ have proper derivatives:

$$\frac{d}{dt} f_p(t,\alpha) \triangleq \int_{-\infty}^{+\infty} \left(\frac{d}{dt} F_p(t+s)\right) dr(s,\alpha) \qquad (4)$$

These are compelling properties for a ***Random Function*** definition – just what we need for studying turbulence.

1 See Wiener "Nonlinear Problems in Random Theory".

2 See Yaglom "Stationary Random Functions", especially section 2.8.

3 See Lumley "Stochastic Tools in Turbulence" especially section 1.6.

4 Yaglom noted that this is a classic "separation of variables" technique.

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Wiener defined two types of polynomials in these integrals, ***Homogeneous Polynomial Functionals*** (similar to a Taylor series) and ***Orthogonal Polynomial Functionals*** (similar to a Hermite Polynomial series). (There are other useful functional expansions techniques, e.g. the Gaussian Transform.)

Homogeneous Polynomial Functionals: The $\mathbf{n^{th}}$ **HPF** term is (using Wiener's notation):

$$u_n(t,\alpha) \triangleq \underbrace{\int\cdots\int U_n(t+s_1,\cdots,t+s_n)\cdot\left(dr(s_1,\alpha)\cdot dr(s_2,\alpha)\cdots dr(s_n,\alpha)\right)}_{\triangleq\mathcal{H}_n\left[U_n(t_1,\cdots,t_n);\alpha\right]} \qquad (5)$$
$$\triangleq \mathcal{H}_n\left[U_n(t_1,\cdots,t_n);\alpha\right]$$

The kernels $\mathbf{U_n(t_1,\cdots,t_n)}$ are symmetrical in the $\mathbf{\{t_n\}}$ arguments. Then the **HPF** expression of $\mathbf{u(t,\alpha)}$ is:

$$u(t,\alpha)=\sum_{p=0}^{\infty}\mathcal{H}_p\left[U_p(t_1,\cdots,t_p);\alpha\right]=\begin{cases}+U_0 & \text{Mean Term}\\ +\mathcal{H}_1\left[U_1(t_1);\alpha\right] & \text{GaussianTerm}\\ +\mathcal{H}_2\left[U_2(t_1,t_2);\alpha\right] & 1^{st}\ \text{non-Gaussian Term}\\ +\mathcal{H}_3\left[U_3(t_1,t_2,t_3);\alpha\right] & 2^{nd}\ \text{non-Gaussian Term}\\ +\ldots & \text{Higher order terms}\end{cases} \qquad (6)$$

These **HPF**'s work well for many flows (e.g. ***Homogeneous Isotropic Turbulence*** – **HIT**). However, there are some convergence issues – analogous to a Taylor Series.

Orthogonal Polynomial Functionals: The **OPF**'s are a complete set of orthogonal (to integration on **α**) functionals which Wiener derived from **HPF**'s by a Gram-Schmidt process. The $\mathbf{n^{th}}$ **OPF** term is (using Wiener's notation) $\mathcal{G}_n[U_n(s_1,s_2,\cdots,s_n);\alpha]$. The $\mathcal{G}_n[\cdots]$ are constructed from the corresponding **HPF** as follows:

$$\begin{aligned}\mathcal{G}_n\left[U_n(t_1,\cdots,t_n);\alpha\right]&=\mathcal{H}_n\left[U_n(t_1,\cdots,t_n);\alpha\right]\\&\quad+C_n^{-2}\mathcal{H}_{n-2}\left[U_{n-2}^{-2}(t_1,\cdots,t_{n-2});\alpha\right]\\&\quad+C_n^{-4}\cdots+C_n^{-6}\cdots+\text{etc}\\U_{n-2}^{-2}(t_1,\cdots,t_{n-2})&=\int U_n(t_1,\cdots,t_{n-2},\lambda,\lambda)d\lambda\\U_{n-2}^{-4}(t_1,\cdots,t_{n-2})&=\iint U_n(t_1,\cdots,t_{n-4},\lambda_1,\lambda_1,\lambda_2,\lambda_2)d\lambda_1 d\lambda_2\end{aligned} \qquad (7)$$

In words, the $\mathbf{n^{th}}$ **OPF** term is constructed from a finite sum of **HPF** terms of order $\mathbf{\{n,n-2,\cdots\}}$ with kernels contracted from the original $\mathbf{U_n(\cdots)}$. The process is straightforward but tedious. (More details are in Wiener **NPRT** and Poduska-62.) Then the **OPF** expansion of $\mathbf{u(t,\alpha)}$ is:

$$u(t,\alpha)=\sum_{p=0}^{\infty}\mathcal{G}_p\left[U_p(t_1,\cdots,t_p);\alpha\right]=\begin{cases}+U_0 & \text{Mean Term}\\ +\mathcal{G}_1\left[U_1(t_1);\alpha\right] & \text{GaussianTerm}\\ +\mathcal{G}_2\left[U_2(t_1,t_2);\alpha\right] & 1^{st}\ \text{non-Gaussian Term}\\ +\mathcal{G}_3\left[U_3(t_1,t_2,t_3);\alpha\right] & 2^{nd}\ \text{non-Gaussian Term}\\ +\ldots & \text{Higher order terms}\end{cases} \qquad (8)$$

These **OPF**'s are more complex and harder to use, but convergence is assured for any stationary flow with locally bounded energy.

The Calculus of Random Functionals: Wiener used this colorful term to describe the toolkit for manipulating his polynomial functionals. Included are formulas for inversion, products, extension to multiple parameters, and much more. Wiener also used the term "***Random Theory***", and Yaglom coined the more descriptive term "***Wiener Machinery***". We use these terms interchangeably.

Extensions to Multiple Parameters: So far we have considered $\mathbf{u(t,\alpha)}$ – a scalar Random Function which is stationary and ergodic on a single parameter "**t**" (not necessarily time). We want to express $\mathbf{u^i(x,y,z,t,\alpha)}$ in polynomial functionals so that the stationarity and ergodicity of the physical situation is captured. For example, for **HIT** in free space, the flow is stationary (i.e. *homogeneous*) and ergodic on all three space dimensions and depends parametrically on **t**, so the appropriate Wiener Integral form is (c.f. (2)):

$$\begin{aligned} f_p(x,y,z,t,\alpha) &\triangleq \int_S F_p(x+x_1, y+y_1, z+z_1, t)\cdot dr(x_1,y_1,z_1,\alpha) \\ &\triangleq \int_S F_p(x^k + x_1^k, t)\cdot dr(x_1^k,\alpha) \end{aligned} \qquad (9)$$

Here the Wiener Process is defined in 3-space and the integral taken over the entire space.

Many steps have been omitted here, and it is no trivial matter to show that this integral exists, but this development can be made quite rigorous. The result is that $\mathbf{f_p(x,y,z,t,\alpha)}$ is ***stationary*** and ***ergodic*** on **{x,y,z}** and depends parametrically on **t**. Moreover, the **HPF**'s and **OPF**'s are defined exactly as before in the single parameter case. This is the basis for our study of **HIT**.

For ***Plane Poiseuille Flow*** (**PPF**) the corresponding Wiener Integral form is:

$$f_p(x,t,z,y,\alpha) \triangleq \int_S F_p(x+x_1, t+t_1, z+z_1, y)\cdot dr(x_1,t_1,z_1,\alpha) \qquad (10)$$

In this case, $\mathbf{f_p(x,t,z,y,\alpha)}$ is ***stationary*** and ***ergodic*** on **{x,t,z}** and depends parametrically on **y**. The **HPF**'s and **OPF**'s are developed analogously.

<u>Extensions to Vector Random Functions:</u> So far we have considered **u(x,y,z,t,α)** – a scalar Random Function which is stationary and ergodic on its parameters in a manner appropriate to the physical situation. But in turbulence we deal with ***vector*** velocities – so it is tempting to simply write $\mathbf{u^i(x,y,z,t,\alpha)}$ and then make vectors of the random functions and kernels. Unfortunately, this is not sufficient. The remedy[5] is to define a vector Wiener Process with independent components from which the Wiener Integral is defined. For **HIT** this is:

$$f_p^i(x,y,z,t,\alpha) \triangleq \int_S F_p^{i\beta}(x^k + x_1^k, t)\cdot dr^{\beta}(x_1^k,\alpha) \qquad (11)$$

The span of the summation on **β** may be more or less than the number of physical dimensions. This is a complex issue and relates to the number of degrees of freedom in the flow. For **HIT** we typically use a span of **3** for symmetry, even though only **2** is required for incompressible flow.

The resolution to this nettlesome matter comes from proving the completeness of the Polynomial Functionals – a subject covered in another short paper.

<u>Summary:</u> This is a short 3-page outline of the Weiner Machinery as we apply it to turbulent flows. Many details have been omitted, but the essence of the argument is this: the Wiener Machinery is a valuable tool uniquely able to describe turbulent fields analytically.

Whenever the polynomial kernels have been determined for a given flow, then all velocity profiles, drag coefficients, energy terms, spectra, etc. can be uniquely calculated. Moreover, precise flow visualizations can be constructed.

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(1026 words)

[5] Waleffe brought this to our attention, and it is examined in great detail in our Chapt04.doc