Progress Report

Applying Wiener Machinery to Turbulence

1. Prologue: What We Want to Accomplish {1021 words}

Almost everyone reading this report will know the history of turbulence, and many will know the history of this research effort. It is safe to skip section 1.1 and read at leisure.

1.1. Turbulence {145 words}

Turbulence — the last unsolved problem of classical physics. This is a progress report on our efforts to apply the Wiener Machinery to ordinary turbulence. The intended audience is Engineers and Physicists more than Mathematicians.

We will cover four basic topics:

1. The Nature of Turbulence – Randomness and Chaos
2. The Tools of Probability – Wiener's Functionals
3. Applying the Wiener Machinery to Turbulence
4. Some Solutions for Specific Flows

We will show that:

1. Turbulent Velocities and Pressures are ordinary real functions.
2. The Wiener Machinery is an effective tool to analyze Turbulence.
3. The Analysis leads to Differential Equations in Deterministic Functions.
4. The Resulting Equations can be solved by ordinary means.

If this process of analysis of turbulence bears up under scrutiny and peer review, then we can declare that a solution to the problem of turbulence has been found.

1.2. A Little History of Turbulence {609 words}

It was Osborne Reynolds who, in 1883, made the first systematic investigation of turbulence. Experimenting with different fluids, pipes, and flow rates, Reynolds found that a single dimensionless parameter is sufficient to define the flow characteristics. This number, known as the Reynolds Number in his honor is defined as:

$$\mathrm{Re} = \frac{\rho V L}{\eta} \qquad (1.2.1)$$

Here, **V** is some characteristic velocity, **L** is some characteristic length, ρ is the mass density, and η is the dynamic viscosity. Reynolds found that, for pipe flow, there is a critical Reynolds Number (in his case about 2100). If the Reynolds Number of the flow were above this critical value, the flow would be ***turbulent*** and chaotic; but if the Reynolds Number of the flow were below this value, the flow would be ***laminar*** and smooth.

Reynolds then carefully analyzed the fundamental Navier-Stokes equations. He assumed that the Navier-Stokes Equations are correct, and he assumed that the solutions have a random as well as a steady component. Reynolds' concept of "random" was primitive at best but his Engineering Intuition was nothing short of brilliant. He averaged the equations, developed the famous "***Reynolds Stresses***" and surely deserves great credit for his fundamental work. We still use the parlance "**RANS**" (Reynolds Averaged Navier-Stokes Equations), "***Reynolds Averaging***" (for ensemble averaging), and "***Reynolds Rules***" (for interchanging averaging operations with derivative and integral operators).

Over the next half century, little theoretical progress was made, but there were some important developments with semi-empirical theories – notably the Logarithmic Velocity Profile. Then in the mid-thirties, Taylor and Batchelor began using statistical methods (such as correlation tensors) until today, some aspects of simple turbulent flows are well understood, at least from the statistical point of view. In the early forties, Kolmogorov presented a set of theories that provide several verifiable predictions about energy transfer in highly symmetric flows (most notably the $\kappa^{-5/3}$ law). Some other techniques (spectral methods, renormalization techniques, etc.) have been applied with marginal success. The fact remains that – aside from this work – no technically interesting flow has ever been satisfactorily analyzed in a theoretical way. The situation is summed up whimsically by Yaglom and Lumley in "A Century of Turbulence, 2001":

> "We believe that, even after 100 years, turbulence studies are still in their infancy. We are naturalists, observing butterflies in the wild. We are still discovering how turbulence behaves, in many respects. We do have a crude, practical, working understanding of many turbulence phenomena, but certainly nothing approaching a comprehensive theory, and nothing that will provide predictions of an accuracy demanded by engineers."

Computers help. Direct Numerical Simulation of Turbulence has been a major arena of activity. Some weak isotropic flows, and some barely turbulent channel flows (Re=3000) have been analyzed. The results are remarkable in character and detail, and they confirm the applicability of the basic Navier-Stokes Equations. The computed velocities show the expected chaotic but ergodic properties. Doubtless, better techniques and improved computer power will remove the limitations of these techniques and we will someday soon have robust Direct Numerical Simulations of many Technically Interesting Flows.

Physicists can compare predicted results with ***experimental*** results only by taking averages. But surely a fully satisfactory ***theoretical*** basis for turbulence must include much more. Surely a robust theory should include a mechanism for constructing ***any*** average or correlation tensor. But it must also include mechanisms for defining the velocities themselves, and all their characteristics, including the derivatives of the velocities. We want to construct specific examples of a typical velocity, which demonstrates the appropriate statistics and derivatives and can be visualized graphically.

Happily, application of Wiener's "***Calculus of Random Functionals***" to Turbulence yields just such results.

1.3. A Little History of This Research Effort {260 words}

Norbert Wiener published a seldom-referenced paper[1] in 1938 suggesting the use of his Random Functional (we now use the term ***Weiner Machinery***) to describe turbulent fluid flows. Later he published the famous book[2] "***Nonlinear Problems in Random Theory***" where in 128 pages he defines;

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[1] N. Wiener, "The Use of Statistical Theory in the Study of Turbulence", 5th International Congress on Applied Mechanics, 1938 (TA350.I61).

[2] N. Wiener, "Nonlinear Problems in Random Theory" MIT Press 1958.

1. The Wiener Process
2. The Wiener Convolution Stochastic Integral
3. Homogeneous Polynomial Functionals
4. Orthogonal Polynomial Functionals
5. Completeness "Proofs"

In fact Wiener laid the foundation for the Wiener Machinery – a set of functional tools more than adequate to the task of analyzing turbulence.

The author published two theses ([JWP-MS-60] and [JWP-ScD-62]) entitled "***Random Theory of Turbulence***" explaining these methods and their application to the analysis of turbulence. Several flows were analyzed and many results presented. The author left the field to pursue a career in computers only to return 30 years later in 1993.

Since 1962 there has been sporadic interest in using the Wiener Machinery to analyze various turbulent flow situations, mostly Isotropic Turbulence looking for the Kolmogorov $\boldsymbol{\kappa}^{-5/3}$ law. These researchers include Yaglom, Meecham, Orszag, Cleever, Saffman, Canavan, and several others. The enthusiasm has been commendable, but the results have been meager. To our knowledge there have been no fully successful analyses of any technically interesting turbulent flows. Indeed, most of these researchers have become discouraged and have abandoned the Wiener Machinery.

The author has returned to the field and reexamined the tools and the results. The results are really very encouraging. This document is a progress report on these renewed efforts.

2. Turbulence and Randomness {698 words}

2.1. The Presumed Nature of Turbulence {378 words}

We adopt several assumptions about turbulence, and we accept the consequences of these assumptions. They are:

1. The fluid is ***Newtonian***, implying a linear symmetric stress tensor.
2. The ***Navier-Stokes Equations*** (in differential form) govern the flow exactly.
3. The flow situations examined are ***Incompressible***.
4. The ***Solutions*** sought are ***Random Functions*** of space and time.
5. The flow situations examined are ***Stationary*** and ***Ergodic*** on at least one parameter.

Newtonian fluids only – we won't deal with ketchup, silly-putty, or coal slurries!

The ***Navier-Stokes Equations*** (in differential form), come from the continuum model. There are other possibilities, including molecular structures, fractal solutions, and integral equations. The fact that Direct Numerical Simulation of the differential equations leads to such good results coupled with the fact that molecular scales are so much smaller than the smallest eddies observed seems to validate the continuum hypothesis. We presume that any solution must be continuous with enough continuous derivatives to allow the equations to be satisfied. The solutions thus have bounded variation and are ***not*** fractals. If we want to include fractals as a possible solution, we will need to (at least) abandon the differential equation form and use a more fundamental (perhaps integral) form of the Navier-Stokes Equations.

Incompressibility must be considered carefully. We must have sufficiently low Mach numbers for physical reality. There are also other effects, including: parabolic vs. hyperbolic equations and reduction in stochastic degrees of freedom.

The ***Solutions*** sought are ***Random Functions*** of space and time. There are other possibilities, including laminar flow, and non-stationary flows. The fact that the Statistical Analyses of Batchelor and Taylor were so successful seems to validate this ensemble hypothesis. There are no observed departures from the ergodic hypothesis, and averages and experiments seem to be quite repeatable.

The flow situations examined are ***Stationary*** and ***Ergodic*** on at least one parameter. There are other possibilities, but for flows that have no apparent stationarity, it may be difficult – if not impossible – to construct an appropriate ensemble. A planar shock wave passing over a sphere might be such an example, or (more simply) a vigorously stirred cup of coffee. We therefore restrict our study to flows which are stationary and ergodic on at least one of **(x,y,z,t)**.

2.2. The Presumed Nature of Randomness in Turbulence {312 words}

What is Turbulence? The easy answer is that ***turbulence*** (in fluids) is the presence of ***random*** velocities, pressures, etc. But this only forces us to ask just what does ***random*** mean? The answer seems so intuitively clear, but just below the surface, the matter becomes complicated and the issues are confusing. In fact, there have been many embarrassingly incorrect statements made about turbulence and randomness in quite recent texts by noted authors, both Physicists and Mathematicians.

At the most fundamental level, by Wiener's definition:

"A Random Physical Quantity (say a turbulent velocity) is a real function of space, time, and a real (sampling) parameter $\alpha \in [0,1]$."

Therefore, a velocity component is just:

$$u(x,y,z,t,\alpha) \qquad (2.2.1)$$

Moreover, the average value of this velocity component is:

$$\mathcal{E}\{u(x,y,z,t,\alpha)\} = \bar{u}(x,y,z,t) = \int_0^1 u(x,y,z,t,\alpha)\cdot d\alpha \qquad (2.2.2)$$

So the parameter α selects a particular path from the sample space, and averages are simple integrals on α over the interval **[0,1]**.

That's about it!

It is not a trivial matter to prove that such a construction exists, but in fact, this can be done for smooth enough random quantities – including all physical quantities in continuum physics. This is the formalism chosen by Wiener[3] and also used by Yaglom[4] and Lumley[5]. It gives us a simple representation of random quantities with all the machinery of the theory of functions of a real variable. More important, we do not have to deal with the burden of measure theory, sigma algebras, etc.

[3] Wiener frequently used this "Steinhaus" model of probability theory. The essence is to consider all random quantities as functions of α versus the traditional notion of a member of an ensemble.

[4] See Yaglom "Stationary Random Functions", especially section 2.8.

[5] See Lumley "Stochastic Tools in Turbulence" especially section 1.6.

There is palpable elegance and pleasing simplicity to this definition of "Randomness". Our tools are now ***functions*** and ***integrals***, versus the standard model with ***sets*** and ***measures***. It is hard to overstate the importance of the view that the ***Random Function*** **v(t,α)** is an ordinary, computable function of its variables **t** and **α**. This places all the "randomness" on **α**, while providing all the tools of real analysis for analyzing **v(t,α)**.

3. The Wiener Machinery In Brief {1718 words}

3.1. The Wiener Process: {161 words}

The ***Wiener Process*** **r(t,α)** is just ***Brownian Motion*** mapped onto the line segment **α∈[0,1]** in such a way that averages are easily represented. Wiener devised a cunning construction of this ***Random Function*** **r(t,α)** which has a simple length measure on **α** which represents "Gaussian Probability." Wiener called **r(t,α)** "***The Random Function of Time and Phase***".(!) These elementary properties are sufficient to completely specify the Wiener Process **r(t,α)**. Most important, this simple process has sufficient power to be the basis of a complete representation of any Random Function.

3.2. The Wiener Integral: {402 words}

We use convolution integrals to express random functions such as **u(t,α)** as follows:

$$f(t,\alpha) \triangleq \int_{-\infty}^{+\infty} F(t-s)\, d_s r(s,\alpha) \qquad (3.2.1)$$

The Random Function **f(t,α)** is the convolution of a smooth function **F(s)**, and the Wiener Process **r(s,α)**. We will plug these integrals into the equations of motion and solve for the **F(s)**.

These Random Functions are ***Stationary*** on **t**, and the **t**-derivative of **f(t,α)** exists:

$$\frac{d}{dt} f(t,\alpha) = \int_a^b \left[\frac{d}{dt} F(t-s)\right] \cdot d_s r(s,\alpha) \qquad (3.2.2)$$

Finally, if the limits are extended to infinity, i.e. **[a,b]=[+∞,-∞]**, then both **f(t,α)** and **g(t,α)** are ***Ergodic*** as well as ***Stationary*** on **t**. In this case, both **F(s)** and **G(s)** have proper Fourier Transforms because they have bounded local variation and are square integrable over infinite limits.

It turns out that sums of products of the **{f_k(t,α)}** form a complete set. This means that any Random Function which is continuous and stationary on **t** and square integrable on **α** can be represented by a sum of products of these **{f_k(t,α)}**. This is our "***completeness theorem***".

Summary: The main result is this: the Wiener Integral defined in equation (4.2.1) provides a mechanism for describing ***Stationary***, perhaps ***Ergodic***, ***Random Functions*** with a well defined derivative and a proper Fourier Transform.

This is a crucial point. We know that the answers we want for turbulent fluid flow are of the form **u(t,α)**; and now, using Wiener Stochastic Integrals, we know how to construct them in terms of ordinary **{F_k(s)}**. If we plug this expansion into the equations of motion, we should get some ordinary, deterministic equations in these **{F_k(s)}**. We have removed the "randomness" from the problem!

We have three important tasks remaining. First, we need to develop the Wiener Machinery with Polynomial Functionals, both Homogeneous and Orthogonal varieties. Second, we need to extend the Wiener Machinery to multiple parameters – space-time. Third, we need to extend the Wiener Machinery to vector Random Functions.

When we have done so, we will show how to tailor the Wiener Process and Integral to various flow situations.

3.3. The Wiener Machinery – Polynomial Functionals: {195 words}

Wiener showed that any scalar Random Function which is continuous and stationary on **t** and square integrable on **α** can be represented by a sum of products of the **{f_k(t,α)}** defined by:

$$f_k(t,\alpha) \triangleq \int_{-\infty}^{+\infty} F_k(t-s)\, d_s r(s,\alpha) \qquad (3.3.1)$$

Here the **{F_k(s)}** are a complete set of functions.

Wiener defined two types of integral polynomial functionals that he called ***Homogeneous Polynomial Functionals*** (**HPF**'s) and ***Orthogonal Polynomial Functionals*** (**OPF**'s). For example, the Random Function **u(t,α)** can be expressed in terms of **HPF**'s as follows:

$$u(t,\alpha) = \left\{ \begin{array}{ll} +U_0(t) & \text{Mean Value} \\ +\int U_1(t-s_1)\cdot d_s r(s_1,\alpha) & 1^{st}\ \text{Term, Gaussian} \\ +\iint U_2(t-s_1,t-s_2) & 2^{nd}\ \text{Term, non-Gaussian} \\ \qquad \cdot d_s r(s_1,\alpha)\cdot d_s r(s_2,\alpha) & \\ +\iiint U_3(t-s_1,t-s_2,t-s_3) & 3^{rd}\ \text{Term, non-Gaussian} \\ \qquad \cdot d_s r(s_1,\alpha)\cdot d_s r(s_2,\alpha)\cdot d_s r(s_3,\alpha) & \\ + \ \cdots \ + & \text{Higher Order Terms} \end{array} \right\} \qquad (3.3.2)$$

In a more condensed form, these **HPF**'s are:

$$\begin{aligned} u(t,\alpha) &= \sum_{k=0}^{\infty} \int \cdots \int U_k(t-s_1,\ldots,t-s_k) \prod_{j=1}^{k} d_s r(s_j,\alpha) \\ &= \sum_{k=0}^{\infty} \mathcal{H}_k\left[U_k(t-s_1,\ldots,t-s_k),\alpha\right] \end{aligned} \qquad (3.3.3)$$

The **$\mathcal{H}_k$[..]** are the **HPF**'s of order **k**. They are **k**-fold Stieltjes Integrals with respect to **r(s,α)** with the kernel **U_k(..)** which is symmetric in its arguments. This **HPF** formulation has many nice properties and is useful in the study of Decaying Turbulent Flows, but it is only weakly convergent – analogous to a Taylor series.

To obtain strongly convergent polynomial functionals, Wiener used Gram-Schmidt orthogonalization to generate ***Orthogonal Polynomial Functionals*** (**OPF**'s) as follows:

$$u(t,\alpha) = \sum_{k=0}^{\infty} \mathcal{G}_k \left[U_k(t-s_1,\ldots,t-s_k);\alpha \right] \tag{3.3.4}$$

These $\mathcal{G}_k$**[..]** are a little harder to use, but they are strongly convergent and can be used very effectively in the study of Turbulent Channel Flows.

3.4. The Wiener Machinery Extended to Multiple Parameters {281 words}

Next, we extend the Polynomial Functionals to the case where **u(x,y,z,t,α)** is stationary on several parameters. For example, in Plane Poiseuille Flow, the velocities are stationary on **(x,z,t)** and the functional formulation for this flow should and will reflect that fact.

Recall the Wiener Integral in one dimension (4.2.1):

$$f(t,\alpha) \triangleq \int_a^b F(t-s)\, d_s r(s,\alpha) \tag{3.4.1}$$

Suppose we try to extend this to two dimensions **(x,y)** as follows:

$$f(x,y,\alpha) \triangleq \int_S F(x-x_1, y-y_1)\cdot d_\sigma r(x_1,y_1,\alpha) \tag{3.4.2}$$

We want this to be done over a surface **S**, with a surface element $\mathbf{d_\sigma r(x,y,\alpha)}$ whose variance is equal to the area of the surface element **ds** covered by $\mathbf{d_\sigma r(x,y,\alpha)}$. If this can be done, then **f(x,y,α)** is stationary and ergodic on both **x** and **y** independently.

Suppose the surface **S** is the entire **(x,y)** plane and let:

$$\begin{aligned} f(x,y,\alpha) &\triangleq \int_S F(x-x_1, y-y_1)\cdot d_\sigma r(x_1,y_1,\alpha) \\ g(x,y,\alpha) &\triangleq \int_S G(x-x_1, y-y_1)\cdot d_\sigma r(x_1,y_1,\alpha) \end{aligned} \tag{3.4.3}$$

Then we get:

$$\begin{aligned} \mathcal{E}\left\{ f(x,y,\alpha)\cdot g(x,y,\alpha) \right\} &\triangleq \int_0^1 f(x,y,\alpha)\cdot g(x,y,\alpha)\cdot d\alpha \\ &= \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{+L} f(\xi,y,\alpha)\cdot g(\xi,y,\alpha)\cdot d\xi \\ &= \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{+L} f(x,\eta,\alpha)\cdot g(x,\eta,\alpha)\cdot d\eta \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x,y)\cdot G(x,y)\cdot dx\, dy \end{aligned} \tag{3.4.4}$$

This is just what we want – namely Random Functions which are independently stationary and ergodic on two parameters represented by the Wiener Integral (4.4.2).

This development can be made quite rigorous, and extension to three or more dimensions is straightforward.

Continuing, Wiener showed that any Random Function which is continuous and stationary on **(x,y)** and square integrable on **α** can be represented by a sum of products of the **$\{f_k(x,y,\alpha)\}$** defined by:

$$f_k(x,y,\alpha) \triangleq \int_S F_k(x-x_1, y-y_1)\, d_\sigma r(x_1,y_1,\alpha) \qquad (3.4.5)$$

The **$\{F_k(x,y)\}$** are a complete set of functions. This means that any Random Function which is continuous and stationary on **(x,y)** and square integrable on **α** can be represented by a sum of products of these **$\{f_k(x,y,\alpha)\}$**. This is our "completeness theorem" over multiple parameters.

Finally, Polynomial functionals (**HPF**'s and **OPF**'s) can be extended to multiple parameters in a straight-forward way.

3.5. The Wiener Machinery Extended to Vectors {504 words}

Suppose now there are several related Random Functions, such as the three velocity components in Plane Poiseuille Flow – how do we handle these with the Wiener Machinery? Specifically, consider a two valued vector of a single parameter: **$u^i(x,\alpha)=[u^1(x,\alpha),u^2(x,\alpha)]$**.

Suppose these two velocity components are closely related, as (say) by the continuity of mass equation:

$$\nabla \cdot \bar{u} = \frac{\partial u^k}{\partial x^k} = \left(\frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} \right) = 0 \qquad (3.5.1)$$

In this case, the two components are ***not*** stochastically independent and either can be derived from the other. Consequently, there is only one "Degree of Freedom", i.e. one independent Random Function, and the scalar formulation above is sufficient.

More generally, there may be only (say) **m** equations inter-relating (say) **n** Random Functions. In this case there are potentially (**n-m**) independent Random Variables which will require an extension of the scalar Wiener Process **$r(t,\alpha)$** to a vector Wiener Process[6] **$r^\beta(t,\alpha)$** where the range of the index **β** is the number of stochastically independent Random Variables (**n-m**).

The Riemann-Stieltjes integral of equation (4.2.1) becomes[7]:

$$\begin{aligned} f_1^i(t,\alpha) &= \int F_1^{i\beta}(t-s)\cdot d_s r^\beta(s,\alpha) \\ f_2^i(t,\alpha) &= \iint F_2^{i\beta_1\beta_2}(t-s_1,t-s_2)\cdot d_s r^{\beta_1}(s_1,\alpha)\cdot d_s r^{\beta_2}(s_2,\alpha) \qquad (3.5.2) \\ &\cdots \text{ etc.} \end{aligned}$$

Here each component of **$d_s r^\beta(t,\alpha)$** for **β=1,2,…** is independent of all others. Then from these integrals, we can construct Homogeneous Polynomial Functionals and Orthogonal Polynomial Functionals as we did for the scalar case. In addition, extension to the multiple parameter case is completely analogous to the treatment above.

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[6] The stochastically independent components of the vector Wiener Process can be defined in terms of the original scalar Wiener Process in several ways.

[7] Note the summation on repeated "**β**" parameters.

This is the most difficult part of extending the Wiener Functionals, because it mixes physical notions of "***Degrees of Freedom***" with mathematical notions of "***Stochastic Dependence***". In many cases, it is unclear just what length of vector Wiener Process we should use. Presumably, the shorter, the better. But a vector length greater than necessary does no harm except to make the resulting equations harder to solve. On the other hand, a vector length less than necessary may yield answers that are a subset of the possibilities provided by a longer vector – such is the case for Isotropic Turbulence. It should be noted that the vector length is not limited to three, for example the case of Isotropic Magneto-Fluid Dynamic Flow might well require a maximum vector length of six!

Fortunately, this complexity usually boils down to the following. For two-dimensional incompressible flows (steady or decaying) there is only one "Degree of Freedom" and the scalar Wiener Process and related functional integrals are appropriate. In three-dimensional incompressible ***steady*** flows, a vector length of two is sufficient and it may well be that the close coupling of the Navier-Stokes equations for this case would reduce the required vector length to one. Recent computational work seems to augur for a 2-vector or even 4-vector **WP** for optimum results. But for three-dimensional incompressible ***decaying*** flows, a vector length of two is sufficient, and is usually required because Navier-Stokes Equations decouple in the final stages of decay. In this latter case, it is convenient to use a vector length of three because of the symmetry this choice allows for the **HPF** Kernels.

In summary, Vector Wiener Processes are required when there is stochastic independence. A short table of the cases considered follows:

All 2D Cases	1-Vector (Scalar) Wiener Process	
3D Poiseuille	2-Vector or 4-Vector Wiener Process	(3.5.3)
Decaying Flows	3-Vector Wiener Process	

3.6. The Wiener Machinery – A Tool for Turbulence {162 words}

The Primary Result of this section 3 is this:

The ***Wiener Machinery*** – Polynomial Functionals based on the Wiener Process, extended to multiple dimensions with multiple parameters – provides a rigorous basis for representing the velocities and other physical quantities of turbulent fluid flow. These representations can be plugged into the equations of motion to yield other equation in the deterministic and square integrable kernels of the functionals. From these, all velocities, pressures, and correlations can be obtained.

It bears repeating that ***Application*** of the Wiener Machinery to a specific flow situation involves several steps:

1. Determine the form of ***Wiener Function*** (the $\mathbf{r}(..,\alpha)$) appropriate to the flow.
2. Determine the number of ***Degrees of Freedom*** to use in the Wiener Function.
3. Determine whether ***Homogeneous Polynomial Functionals*** (**HPF**'s), ***Orthogonal Polynomial Functionals*** (**OPF**'s) or some other form (such as the ***Gaussian Transform*** discussed elsewhere)

Such is the hard work of using the Wiener Machinery.

4. The Wiener Machinery In Full {1718 words}

4.1. The Wiener Process: {161 words}

The ***Wiener Process*** **r(t,α)** is just ***Brownian Motion*** mapped onto the line segment **α∈[0,1]** in such a way that averages are easily represented. Wiener devised a cunning construction of this ***Random Function*** **r(t,α)** which has a simple length measure on **α** which represents "Gaussian Probability." Wiener called **r(t,α)** "***The Random Function of Time and Phase***".(!)

The resulting ***Random Function*** **r(t,α)** has some interesting properties. With regard to **t**, for every (emphasize ***every***, not ***almost every***) value of **α**, **r(t,α)** is: continuous on **t**, not bounded in variation, and a fractal with fractal dimension **3/2**. With regard to **α**, for every value of **t**, **r(t,α)** is integrable, and square integrable. Moreover, **r(t,α)** has increments which are Gaussian distributed and are independent for non-overlapping intervals.

These elementary properties are sufficient to completely specify the Wiener Process **r(t,α)**. Most important, this simple process has sufficient power to be the basis of a complete representation of any Random Function.

4.2. The Wiener Integral: {402 words}

We express random functions such as **u(t,α)** by integrals involving the Wiener Process. The basis for such an expansion is this Riemann-Stieltjes integral[8]:

$$f(t,\alpha) \triangleq \int_a^b F(t-s)\, d_s r(s,\alpha) \qquad (4.2.1)$$

The Random Function **f(t,α)** is the convolution of a smooth function **F(s)**, and the Wiener Process **r(s,α)**. We will plug these integrals into the equations to be solved; convert these equations to deterministic equations in **F(s)**; then solve the resulting equations by conventional means – numerically if necessary.

Now extend this integral to a complete set of functions **{F_k(s)}** to obtain:

$$f_k(t,\alpha) \triangleq \int_a^b F_k(t-s)\, d_s r(s,\alpha) \qquad (4.2.2)$$

The properties of this Convolution Integral are just the ones we need to study turbulence. Assuming all kernel functions are periodic beyond **[a,b]**, define:

$$f(t,\alpha) = \int_a^b F(t-s)\, d_s r(s,\alpha), \quad g(t,\alpha) = \int_a^b G(t-s)\, d_s r(s,\alpha) \qquad (4.2.3)$$

One can quickly show that these Random Functions are ***Stationary*** on **t**. Moreover, and of great importance, the **t**-derivative of **f(t,α)** exists and is:

$$\frac{d}{dt} f(t,\alpha) = \int_a^b \left[\frac{d}{dt} F(t-s)\right] \cdot d_s r(s,\alpha) \qquad (4.2.4)$$

[8] Yaglom noted that this is a classic "separation of variables" technique.

Finally, if the limits are extended to infinity, i.e. **[a,b]=[+∞,-∞]**, then both **f(t,α)** and **g(t,α)** are ***Ergodic*** as well as ***Stationary*** on **t**. In this case, both **F(s)** and **G(s)** have proper Fourier Transforms because they have bounded local variation and are square integrable over infinite limits.

It turns out that sums of products of the **{f_k(t,α)}** form a complete set. This means that any Random Function which is continuous and stationary on **t** and square integrable on **α** can be represented by a sum of products of these **{f_k(t,α)}**. This is our "***completeness theorem***".

Summary: The main result is this: the Wiener Integral defined in equation (4.2.1) provides a mechanism for describing ***Stationary***, perhaps ***Ergodic***, ***Random Functions*** with a well defined derivative and a proper Fourier Transform.

This is a crucial point. We know that the answers we want for turbulent fluid flow are of the form **u(t,α)**; and now, using Wiener Stochastic Integrals, we know how to construct them in terms of ordinary **{F_k(s)}**. If we plug this expansion into the equations of motion, we should get some ordinary, deterministic equations in these **{F_k(s)}**. We have removed the "randomness" from the problem!

We have three important tasks remaining. First, we need to develop the Wiener Machinery with Polynomial Functionals, both Homogeneous and Orthogonal varieties. Second, we need to extend the Wiener Machinery to multiple parameters – space-time. Third, we need to extend the Wiener Machinery to vector Random Functions.

When we have done so, we will show how to tailor the Wiener Process and Integral to various flow situations.

4.3. The Wiener Machinery – Polynomial Functionals: {195 words}

Wiener showed that any scalar Random Function which is continuous and stationary on **t** and square integrable on **α** can be represented by a sum of products of the **{f_k(t,α)}** defined by:

$$f_k(t,\alpha) \triangleq \int_{-\infty}^{+\infty} F_k(t-s)\, d_s r(s,\alpha) \qquad (4.3.1)$$

Here the **{F_k(s)}** are a complete set of functions.

Wiener defined two types of integral polynomial functionals that he called ***Homogeneous Polynomial Functionals*** (**HPF**'s) and ***Orthogonal Polynomial Functionals*** (**OPF**'s). For example, the Random Function **u(t,α)** can be expressed in terms of **HPF**'s as follows:

$$u(t,\alpha) = \left\{ \begin{array}{ll} +U_0(t) & \text{Mean Value} \\ +\int U_1(t-s_1)\cdot d_s r(s_1,\alpha) & 1^{st}\ \text{Term, Gaussian} \\ +\iint U_2(t-s_1, t-s_2) & 2^{nd}\ \text{Term, non-Gaussian} \\ \qquad \cdot d_s r(s_1,\alpha)\cdot d_s r(s_2,\alpha) & \\ +\iiint U_3(t-s_1, t-s_2, t-s_3) & 3^{rd}\ \text{Term, non-Gaussian} \\ \qquad \cdot d_s r(s_1,\alpha)\cdot d_s r(s_2,\alpha)\cdot d_s r(s_3,\alpha) & \\ + \ \cdots \ + & \text{Higher Order Terms} \end{array} \right\} \qquad (4.3.2)$$

In a more condensed form, these **HPF**'s are:

$$\begin{aligned} u(t,\alpha) &= \sum_{k=0}^{\infty} \int \cdots \int U_k(t-s_1,\ldots,t-s_k) \prod_{j=1}^{k} d_s r(s_j,\alpha) \\ &= \sum_{k=0}^{\infty} \mathcal{H}_k \left[U_k(t-s_1,\ldots,t-s_k), \alpha \right] \end{aligned} \qquad (4.3.3)$$

The $\mathcal{H}_k$**[..]** are the **HPF**'s of order **k**. They are **k**-fold Stieltjes Integrals with respect to **r(s,α)** with the kernel **U_k(..)** which is symmetric in its arguments. This **HPF** formulation has many nice properties and is useful in the study of Decaying Turbulent Flows, but it is only weakly convergent – analogous to a Taylor series.

To obtain strongly convergent polynomial functionals, Wiener used Gram-Schmidt orthogonalization to generate ***Orthogonal Polynomial Functionals*** (**OPF**'s) as follows:

$$u(t,\alpha) = \sum_{k=0}^{\infty} \mathcal{G}_k \left[U_k(t-s_1,\ldots,t-s_k); \alpha \right] \qquad (4.3.4)$$

These $\mathcal{G}_k$**[..]** are a little harder to use, but they are strongly convergent and can be used very effectively in the study of Turbulent Channel Flows.

4.4. The Wiener Machinery Extended to Multiple Parameters {281 words}

Next, we extend the Polynomial Functionals to the case where **u(x,y,z,t,α)** is stationary on several parameters. For example, in Plane Poiseuille Flow, the velocities are stationary on **(x,z,t)** and the functional formulation for this flow should and will reflect that fact.

Recall the Wiener Integral in one dimension (4.2.1):

$$f(t,\alpha) \triangleq \int_a^b F(t-s)\, d_s r(s,\alpha) \qquad (4.4.1)$$

Suppose we try to extend this to two dimensions **(x,y)** as follows:

$$f(x,y,\alpha) \triangleq \int_S F(x-x_1, y-y_1) \cdot d_\sigma r(x_1,y_1,\alpha) \qquad (4.4.2)$$

We want this to be done over a surface **S**, with a surface element **d_σr(x,y,α)** whose variance is equal to the area of the surface element **ds** covered by **d_σr(x,y,α)**. If this can be done, then **f(x,y,α)** is stationary and ergodic on both **x** and **y** independently.

Suppose the surface **S** is the entire **(x,y)** plane and let:

$$\begin{aligned} f(x,y,\alpha) &\triangleq \int_S F(x-x_1, y-y_1) \cdot d_\sigma r(x_1,y_1,\alpha) \\ g(x,y,\alpha) &\triangleq \int_S G(x-x_1, y-y_1) \cdot d_\sigma r(x_1,y_1,\alpha) \end{aligned} \qquad (4.4.3)$$

Then we get:

$$\begin{aligned}\mathcal{E}\{f(x,y,\alpha)\cdot g(x,y,\alpha)\} &\triangleq \int_0^1 f(x,y,\alpha)\cdot g(x,y,\alpha)\cdot d\alpha \\ &= \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{+L} f(\xi,y,\alpha)\cdot g(\xi,y,\alpha)\cdot d\xi \\ &= \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{+L} f(x,\eta,\alpha)\cdot g(x,\eta,\alpha)\cdot d\eta \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x,y)\cdot G(x,y)\cdot dx\,dy\end{aligned} \tag{4.4.4}$$

This is just what we want – namely Random Functions which are independently stationary and ergodic on two parameters represented by the Wiener Integral (4.4.2).

This development can be made quite rigorous, and extension to three or more dimensions is straightforward.

Continuing, Wiener showed that any Random Function which is continuous and stationary on **(x,y)** and square integrable on **α** can be represented by a sum of products of the **$\{f_k(x,y,\alpha)\}$** defined by:

$$f_k(x,y,\alpha) \triangleq \int_S F_k(x-x_1,y-y_1)\,d_\sigma r(x_1,y_1,\alpha) \tag{4.4.5}$$

The **$\{F_k(x,y)\}$** are a complete set of functions. This means that any Random Function which is continuous and stationary on **(x,y)** and square integrable on **α** can be represented by a sum of products of these **$\{f_k(x,y,\alpha)\}$**. This is our "completeness theorem" over multiple parameters.

Finally, Polynomial functionals (**HPF**'s and **OPF**'s) can be extended to multiple parameters in a straightforward way.

4.5. The Wiener Machinery Extended to Vectors {504 words}

Suppose now there are several related Random Functions, such as the three velocity components in Plane Poiseuille Flow – how do we handle these with the Wiener Machinery? Specifically, consider a two valued vector of a single parameter: **$u^i(x,\alpha)=[u^1(x,\alpha),u^2(x,\alpha)]$**.

Suppose these two velocity components are closely related, as (say) by the continuity of mass equation:

$$\nabla\cdot\bar{u} = \frac{\partial u^k}{\partial x^k} = \left(\frac{\partial u^1}{\partial x^1}+\frac{\partial u^2}{\partial x^2}\right) = 0 \tag{4.5.1}$$

In this case, the two components are ***not*** stochastically independent and either can be derived from the other. Consequently, there is only one "Degree of Freedom", i.e. one independent Random Function, and the scalar formulation above is sufficient.

More generally, there may be only (say) **m** equations inter-relating (say) **n** Random Functions. In this case there are potentially (**n-m**) independent Random Variables which will require an extension of the scalar

Wiener Process $\mathbf{r(t,\alpha)}$ to a vector Wiener Process[9] $\mathbf{r^{\beta}(t,\alpha)}$ where the range of the index $\boldsymbol{\beta}$ is the number of stochastically independent Random Variables (**n-m**).

The Riemann-Stieltjes integral of equation (4.2.1) becomes[10]:

$$
\begin{aligned}
f_1^i(t,\alpha) &= \int F_1^{i\beta}(t-s)\cdot d_s r^{\beta}(s,\alpha) \\
f_2^i(t,\alpha) &= \iint F_2^{i\beta_1\beta_2}(t-s_1,t-s_2)\cdot d_s r^{\beta_1}(s_1,\alpha)\cdot d_s r^{\beta_2}(s_2,\alpha) \\
&\cdots \text{ etc.}
\end{aligned}
\qquad (4.5.2)
$$

Here each component of $\mathbf{d_s r^{\beta}(t,\alpha)}$ for $\boldsymbol{\beta=1,2,\ldots}$ is independent of all others. Then from these integrals, we can construct Homogeneous Polynomial Functionals and Orthogonal Polynomial Functionals as we did for the scalar case. In addition, extension to the multiple parameter case is completely analogous to the treatment above.

This is the most difficult part of extending the Wiener Functionals, because it mixes physical notions of "***Degrees of Freedom***" with mathematical notions of "***Stochastic Dependence***". In many cases, it is unclear just what length of vector Wiener Process we should use. Presumably, the shorter, the better. But a vector length greater than necessary does no harm except to make the resulting equations harder to solve. On the other hand, a vector length less than necessary may yield answers that are a subset of the possibilities provided by a longer vector – such is the case for Isotropic Turbulence. It should be noted that the vector length is not limited to three, for example the case of Isotropic Magneto-Fluid Dynamic Flow might well require a maximum vector length of six!

Fortunately, this complexity usually boils down to the following. For two-dimensional incompressible flows (steady or decaying) there is only one "Degree of Freedom" and the scalar Wiener Process and related functional integrals are appropriate. In three-dimensional incompressible ***steady*** flows, a vector length of two is sufficient and it may well be that the close coupling of the Navier-Stokes equations for this case would reduce the required vector length to one. Recent computational work seems to augur for a 2-vector or even 4-vector **WP** for optimum results. But for three-dimensional incompressible ***decaying*** flows, a vector length of two is sufficient, and is usually required because Navier-Stokes Equations decouple in the final stages of decay. In this latter case, it is convenient to use a vector length of three because of the symmetry this choice allows for the **HPF** Kernels.

In summary, Vector Wiener Processes are required when there is stochastic independence. A short table of the cases considered follows:

All 2D Cases	1-Vector (Scalar) Wiener Process	
3D Poiseuille	2-Vector or 4-Vector Wiener Process	(4.5.3)
Decaying Flows	3-Vector Wiener Process	

4.6. The Wiener Machinery – A Tool for Turbulence {162 words}

The Primary Result of this section 3 is this:

The ***Wiener Machinery*** – Polynomial Functionals based on the Wiener Process, extended to multiple dimensions with multiple parameters – provides a rigorous basis for representing the velocities and other physical

[9] The stochastically independent components of the vector Wiener Process can be defined in terms of the original scalar Wiener Process in several ways.
[10] Note the summation on repeated "$\boldsymbol{\beta}$" parameters.

quantities of turbulent fluid flow. These representations can be plugged into the equations of motion to yield other equation in the deterministic and square integrable kernels of the functionals. From these, all velocities, pressures, and correlations can be obtained.

It bears repeating that ***Application*** of the Wiener Machinery to a specific flow situation involves several steps:

1. Determine the form of ***Wiener Function*** (the **r(..,α)**) appropriate to the flow.
2. Determine the number of ***Degrees of Freedom*** to use in the Wiener Function.
3. Determine whether ***Homogeneous Polynomial Functionals*** (**HPF**'s), ***Orthogonal Polynomial Functionals*** (**OPF**'s) or some other form (such as the ***Gaussian Transform*** discussed elsewhere)

Such is the hard work of using the Wiener Machinery.

5. Overview – Solutions for Two Specific Flows

This section is an overview of the solution process – how do we apply the Wiener Machinery to Turbulent Fluid Flows. Sections 5 and 6 give details for Isotropic Turbulence and Plane Poiseuille Flow respectively.

The intent of this chapter is not to describe these results is complete detail. The intent is to show that the Wiener Machinery can be used to analyze some very interesting turbulent flows.

5.1. Applying Wiener Functionals to Turbulence {347 words}

In section 4.3, we showed that the velocities of turbulent flow can be expressed in terms of ***Homogeneous Polynomial Functionals*** (**HPF**'s) and ***Orthogonal Polynomial Functionals*** (**OPF**'s) suitably structured to reflect the stationarity of the flow. How can they be applied?

First, let us verify that the tools fit the job: From the continuum hypothesis and the differential equation form of the equations of motion (section 2 above) the velocities must be continuous and sufficiently differentiable to satisfy the equations of motion. Moreover, the velocities must have bounded variance, because energy per unit mass is bounded. These two constraints – continuity and bounded energy – are sufficient to insure that the Wiener Polynomial Expansions of turbulent velocities exist and converge.

So we expand the $\mathbf{u^i(x,y,z,t,\alpha)}$ in terms of Wiener Functionals – **HPF**'s or **OPF**'s suitably structured to reflect the stationarity of the flow. Next, we plug these functional expansions into the equations of motion. The functionals are unique and invertible, so the equations of motion in the Random Functions $\mathbf{u^i(x,y,z,t,\alpha)}$ become a sequence of coupled equations in the Kernels $\mathbf{U_k^i(..)}$. The ***salient*** point is this: the resulting equations are deterministic! The randomness has been removed from the problem.

The remaining problem is now to solve for the Kernels. In some cases, this can be done by direct analysis, in other cases, approximate methods and even numerical methods must be employed. But once the solutions are obtained, any characteristic of that particular flow can be calculated: any correlation, any average, any visualization, anything.

Some might argue that a countable set of equations in the Kernels is no better than a countable set of equations from any other "closure" method. But our sequence of Polynomial Functionals converges, hence the higher order functionals must decrease in size and we can approximate the velocities, pressures, and any other physical quantity and/or correlation to any desired accuracy with a finite sum of functionals.

In short, we have converted the equations of motion from stochastic equations that are intractable to a collection of deterministic equations that are imminently solvable.

5.2. The Two Specific Cases {205 words}

There are many interesting turbulent flow situations worthy of analysis. Two broad classes are ***Decaying Unbounded Flows*** and ***Steady Channel Flows***. Each of these cases has an Ordinary Hydrodynamic Case and a Magneto-Hydrodynamic Case. These four broad classes have been studied in detail by the author and others. We report here on two of these studies.

Among the decaying flows, we choose ***Decaying Isotropic Turbulence*** (aka **HIT** – Homogeneous Isotropic Turbulence). This well-studied flow has little technical importance, but does provide a convenient and simplified basis to investigate. Our formal study provides an example of the use of ***Homogeneous Polynomial Functionals***, and leads to some novel results.

Among the steady flows, we choose ***Plane Poiseuille Flow*** as the exemplar. This flow has great technical importance and has resisted all theoretical assaults. The best that one could do theoretically before the Wiener Functionals is the Logarithmic Velocity Profile that relies on experiment for constants and does not satisfy the boundary conditions. Our study provides an example of the use of ***Orthogonal Polynomial Functionals***, and leads to a convergent sequence approximation to the velocities. We are presently able to predict velocity profiles of the steady and random components with reasonable accuracy.

5.3. Decaying Isotropic Turbulence - HIT {618 words}

This classic flow is stationary on three space dimensions **(x,y,z)** and decaying in time. We expand the velocities in the appropriate ***Homogeneous Polynomial Functionals***[11], and we look for solutions that have Isotropic Correlation Tensors and are "lowest order" that is, slowest decaying. This analysis is essentially a perturbation method applied to the final stages of decay (the "Dissipation Range"). We are looking for asymptotic characteristics such as decay laws, stability of isotropy, higher order correlations, etc. It would be overly optimistic to expect these results to extend (i.e., converge strongly) to the more energetic regions (the "Inertial Range"). For that we will likely need the better convergent Orthogonal Polynomial Functionals. It fortunately turns out that the equations for the Kernels can be solved in sequence. Moreover, the Kernels are square integrable in their own (infinite) domains and therefore have ***proper*** Fourier Transforms.

So doing, we solve the equations of motion in sequence, and achieve the following:

1. Calculate 1st, 2nd, and 3rd order Velocities and pressure
2. The 1st Order **p** is zero (we usually can't hear it)
3. The 1st order **2-V** correlation tensor decays as $\mathbf{t}^{-5/2}$ (or $\mathbf{t}^{-3/2}$, see below)
4. The 1st order **2-V** correlation tensor allows an alternator with $\mathbf{t}^{-3}$ decay
5. The 2nd order **p** is not zero! (We can hear it when it's really strong!)
6. The 2nd order **2-V** correlation tensor decays as $\mathbf{t}^{-4}$ **(**or $\mathbf{t}^{-2}$, see below)?
7. Calculate the **3-V** correlation tensor decays as $\mathbf{t}^{-9/2}$ **(**or $\mathbf{t}^{-5/2}$, see below)?

[11] We use a vector Wiener Process of length three.

Conclusions: Any order velocity, pressure, and/or correlation tensor can be calculated, simulated and visualized using the Wiener Machinery. Does this qualify as a closed form solution to the final stages of Decaying Isotropic Turbulence? I think so.

Some Comments:

1. All orders of velocities and pressure can be calculated and related to the lowest order (Gaussian) terms. For example, the three-velocity correlation tensor can be related directly to the two-velocity correlation tensor.
2. The asymptotic decay law is a power law of **t**, fundamentally $\mathbf{t}^{-5/2}$ but (as Saffman suggested) $\mathbf{t}^{-3/2}$ is possible. In fact many other (faster) decay laws are possible, depending on initial conditions.
3. The **2-V** correlation tensor allows (but does not require) an alternator term. This should come as no surprise, but it is often omitted in the literature[12].
4. For Isotropic Turbulence in the final stages of decay, the effective Reynolds Number is quite low, and we should expect no Kolmogorov $\boldsymbol{\kappa}^{-5/3}$ decay law. In other words, these results don't extend up into the inertial range.
5. There exist solutions which consist of even order functionals only, i.e. the Gaussian (and all other odd order functionals) are missing! These solutions are stable and should be present in numerical simulations with suitable initial conditions.

Finally, other non-isotropic decaying flows do ***not*** seem to approach isotropy in their final stages of decay. In fact, we have some preliminary results to the contrary. Consider a field of Isotropic Turbulence in its final stages of decay. Suppose there is a magnetic field present, and that at time $\mathbf{T_1}$, electrical conductivity gets turned on (perhaps by temperature change or radiation) – the velocity field becomes Axisymmetric. Suppose that at a later time $\mathbf{T_2}$, conductivity gets turned off (temperature drops or radiation ceases) – does the axisymmetric velocity field revert to an Isotropic field? Our early results say No! This implies that Isotropic Turbulence is ***not*** the unique natural end-state of turbulence.

There is much more to be said and done here. We have also analyzed the axisymmetric decaying turbulence case (as in a wind tunnel with a wire gauze generator) and have some early results which need to be expanded and recorded.

5.4. Plane Poiseuille Flow {563 words}

This simple flow, which has great Engineering significance, is stationary on **(x,z,t)** the stream-wise dimension, the span-wise dimension, and time. We expand the velocities in the appropriate ***Orthogonal Polynomial Functionals*** and insert the expansion into the equations of motion, which then yield a countable set of non-linear partial differential equations which are deterministic and square integrable over their domains. The **OPF** (vs. the **HPF**) formulation is required for this flow because of the strong convergence provided by the **OPF**'s.

We know of no direct solution to these partial differential equations, but we do know that the Kernels can be expanded in Hermite Functions on **(x,z,t)** and the resulting ordinary differential equations in **y** are then a countable coupled set. There are many methods to optimize a sequence of approximations to the complete equations including least square error, Galerkin methods, and the Ritz Method. There are also a host of symmetry conditions to apply to the expansions that greatly reduces the parameter count.

Our preferred method – which is certainly not unique – is to use a ***Galerkin Error Residual Method*** (a '**GERM**' method) to define an error function which measures the "goodness" of an approximation, and then min-

[12] Actually Batchelor mentions the possibility, see his equation (3.3.11)

imize this error with respect to all free parameters. We have done this in full three dimensions for a sequence of Reynolds Numbers (mean bulk flow velocity) ranging from 1000 to 10,000,000 with excellent results.

We note the following favorable points of comparison with known forms and experimental results for this flow:

1. The Mean Velocity in mid-channel is quadratic and roughly independent of viscosity; while near the wall it is linear, proportional to wall stress and roughly independent of channel width.
2. The Reynolds Stresses $\mathcal{E}\{\mathbf{u_x u_y}\}$ are linear over mid channel and have a multiple order zero at the wall.
3. The stream-wise random component $\mathcal{E}\{\mathbf{u_x^2}\}$ is slow changing in mid-channel, has a high peak near the wall, and has a double order zero at the wall.
4. The cross-channel random component $\mathcal{E}\{\mathbf{u_y^2}\}$ is slow changing in mid-channel, has no high peak near the wall, and has a fourth order zero at the wall.
5. The friction factor closely follows the shape of the Colebrook formula, but gives somewhat lower results – i.e. less drag.

There is still much work to be done on this flow, including time and space displaced correlation tensors, use of higher order functionals, visualization of solutions, etc. But results to date are extremely encouraging. To our knowledge, this is the first solution to Plane Poiseuille Flow that does not involve use of experimental measurements or additional physical hypotheses.

Some Comments:

1. The random components of velocity are very nearly Gaussian, yet the higher order terms are essential to limit growth.
2. The van Kármán "Velocity Defect Law" and the Prandtl "Wall Velocity Law" are validated both theoretically and computationally.
3. Finally, this analysis will be refined with better expansion techniques, extension to the Magneto-Hydrodynamic Case, and use of other methods, such as the Gaussian Transform. In addition, we may extend the analysis to Circular Poiseuille Flow because of its practical importance and the better availability of experimental results.

Figure x.01 below shows the mean velocity profile and Reynolds Stresses for $\mathbf{R_e=40000}$, based on mean velocity and channel half-width (equivalent to a hydraulic Reynolds Number $\mathbf{R_{hd}=160000}$). Also attached is a visualization loop, computed directly from the GERM results.

6. Details of Isotropic Turbulence

This section contains a moderate amount of detail about solving the equations of motion for Isotropic Turbulence using Wiener's ***Homogeneous Polynomial Functionals***. But the subject is not covered completely here. A complete description is contained in Chapters 7-11 of the monograph.

6.1. Details of Basis {257 words}

Isotropic Turbulence is stationary on three space dimensions **(x,y,z)** and decays in time. We therefore choose $\mathbf{r(x,y,z,\alpha)}$ as the appropriate Wiener Process – notice that **t** is not included[13]. The vector Wiener Integral is then:

[13] Meecham and his colleagues (q.v.) tried to improve convergence using a "time-varying" base.

$$u_1^i(x^k,t,\alpha) = \int U_1^{i\beta}(x+x_1,y+y_1,z+z_1,t)\, d_{xyz} r^\beta(x_1,y_1,z_1,\alpha) = \int U_1^{i\beta}(x^k+x_1^k,t)\, d_{x_1^k} r^\beta(x_1^k,\alpha) \quad (6.1.1)$$

We choose to sum **β** over **β=1,2,3** even though a range of **β=1,2** would be sufficient[14].

We now choose ***Homogeneous Polynomial Functionals*** to expand the $\mathbf{u^i(x,y,z,t,\alpha)}$ using the Wiener Integral defined above. This analysis is essentially a perturbation method applied to the final stages of decay (the "Dissipation Range"). The first several terms are then:

$$u^i(x^k,t,\alpha) = +\int U_1^{i\beta}(x^k+x_1^k,t)\, d_s r^\beta(x_1^k,\alpha) + \iint U_2^{i\beta_1\beta_2}(x^k+x_1^k,x^k+x_2^k,t)\, d_s r^{\beta_1}(x_1^k,\alpha)\, d_s r^{\beta_2}(x_2^k,\alpha) + \text{etc.} \quad (6.1.2)$$

Note that we have omitted a "mean" term which must be justified. Also note that each of the $\mathbf{U_k(..)}$ has a proper Fourier transform. The equivalent "functional notation shorthand" is:

$$u^i(s,t,\alpha) = \sum_{n=1}^{\infty} \mathcal{H}_n\left[U_n^{i\beta_{\$n}}(s_{\$n},t);\alpha\right] = \left\{\begin{array}{ll} +\mathcal{H}_1\left[U_1^{i\beta_1}(s_1,t);\alpha\right] & \text{First Term, Gaussian} \\ +\mathcal{H}_2\left[U_2^{i\beta_1\beta_2}(s_1,s_2,t);\alpha\right] & \text{Second Term} \\ +\text{etc.} & \text{Higher Order Terms} \end{array}\right\} \quad (6.1.3)$$

The notation **\$n** means a sequence, e.g. $\mathbf{x_{\$3}=x_1,x_2,x_3}$ – a device taken from Maple.

We are looking for solutions which are Isotropic. This means that all averages are invariant to translations and rotations in space. We do not assume invariant to reflections, since neither the physics nor the math require it. As we shall see, the alternator term is allowed, but not required. We therefore choose to use several isotropic correlation tensors as follows:

$$\begin{aligned} L^i(\xi^k,t) &= \mathcal{F}_1\left[\Lambda^i(\kappa^k,t)\right] = \mathcal{E}\left\{p(x^k,t;\alpha)\cdot u^i(x^k+\xi^k,t;\alpha)\right\} \\ R^{ij}(\xi^k,t) &= \mathcal{F}_1\left[\Phi^{ij}(\kappa^k,t)\right] = \mathcal{E}\left\{u^i(x^k,t;\alpha)\cdot u^j(x^k+\xi^k,t;\alpha)\right\} \\ S^{ijl}(\xi^k,t) &= \mathcal{F}_1\left[\Sigma^{ijl}(\kappa^k,t)\right] = \mathcal{E}\left\{u^i(x^k,t;\alpha)\cdot u^j(x^k,t;\alpha)\cdot u^l(x^k+\xi^k,t;\alpha)\right\} \end{aligned} \quad (6.1.4)$$

These are respectively, the "Two-Point, Pressure-Velocity" (**2PPV**), the "Two-Point, Two-Velocity" (**2P2V**), and the "Two-Point, Three-Velocity" (**2P3V**) Correlation Tensors. For Isotropic invariance[15], these tensors must have the form:

---

[14] It turns out that a single **β=1** yields all essential results, with much less complexity, but requiring (versus allowing) the alternator term.

[15] Batchelor [Batch-53] gives an excellent account of this theory developed by H.P. Robertson in 1940.

$$
\begin{aligned}
&L^i(\xi^k,t) = L_p(\bar{\xi},t)\,\xi^i \\
&\Lambda^i(\kappa^k,t) = \Lambda_p(\bar{\kappa},t)\,\kappa^i \\
&R^{ij}(\xi^k,t) = R_p(\bar{\xi},t)\,\xi^i\xi^j + R_q(\bar{\xi},t)\,\delta^{ij} + R_r(\bar{\xi},t)\,\varepsilon^{ijk}\xi^k \\
&\Phi^{ij}(\kappa^k,t) = \Phi_p(\bar{\kappa},t)\,\kappa^i\kappa^j + \Phi_q(\bar{\kappa},t)\,\delta^{ij} + \Phi_r(\bar{\kappa},t)\,\varepsilon^{ijk}\kappa^k \\
&S^{ijl}(\xi^k,t) = S_p(\bar{\xi},t)\,\xi^i\xi^j\xi^l + S_q(\bar{\xi},t)\cdot(\xi^i\delta^{jl} + \xi^j\delta^{il}) + S_r(\bar{\xi},t)\,\delta^{ij}\xi^l \\
&\qquad + S_s(\bar{\xi},t)\cdot(\xi^i\varepsilon^{jlk}\xi^k + \xi^j\varepsilon^{ilk}\xi^k) \\
&\Sigma^{ijl}(\kappa^k,t) = \Sigma_p(\bar{\kappa},t)\,\kappa^i\kappa^j\kappa^l + \Sigma_q(\bar{\kappa},t)\cdot(\kappa^i\delta^{jl} + \kappa^j\delta^{il}) + \Sigma_r(\bar{\kappa},t)\,\delta^{ij}\kappa^l \\
&\qquad + \Sigma_s(\bar{\kappa},t)\cdot(\kappa^i\varepsilon^{jlk}\kappa^k + \kappa^j\varepsilon^{ilk}\kappa^k)
\end{aligned}
\tag{6.1.5}
$$

Focusing on the Fourier Transforms expanded in $\boldsymbol{\kappa}$, Mass-Conservation requires:

$$
\begin{aligned}
&\Lambda^i(\kappa^k,t) = \Lambda_a(\bar{\kappa},t)\,\kappa^i = 0 \\
&\Phi^{ij}(\kappa^k,t) = \Phi_a(\bar{\kappa},t)\cdot\Pi^{ij}(\kappa^k) + \Phi_b(\bar{\kappa},t)\cdot\varepsilon^{ijk}\kappa^k \\
&\Sigma^{ijl}(\kappa^k,t) = \Sigma_a(\bar{\kappa},t)\cdot\Gamma^{lij}(\kappa^k) + \Sigma_b(\bar{\kappa},t)\cdot\left(\kappa^i\varepsilon^{jlk}\kappa^k + \kappa^j\varepsilon^{ilk}\kappa^k\right)
\end{aligned}
\tag{6.1.6}
$$

Here, the two projection operators are defined as:

$$
\Pi^{ij}(\kappa^k) = \left(\delta^{ij} - \frac{\kappa^i\kappa^j}{\bar{\kappa}^2}\right) \quad \text{and} \quad \Gamma^{ipq}(\kappa^k) = \left(\kappa^i\,\frac{\kappa^p\kappa^q}{\bar{\kappa}^2} - \frac{\kappa^q\delta^{ip} + \kappa^p\delta^{iq}}{2}\right)
\tag{6.1.7}
$$

These are the tools and definitions we need to attack Isotropic Turbulence.

6.2. Details of Solution Procedure {183 words}

The normalized Navier-Stokes equations have only one dimensionless parameter – the ubiquitous Reynolds Number. But there is no a priori Time, Length, or Velocity scale in the Isotropic case, so we elect to solve the equations of motion in ***physical*** units. Fortunately, there is only one constitutive parameter, namely $\boldsymbol{\nu}$ the viscosity. Thus the equations we solve are:

$$
\begin{aligned}
&\left(\partial_t - \nu\nabla^2\right)u^i(x^k,t,\alpha) - \frac{\partial}{\partial x^i}\,p(x^k,t,\alpha) = -\frac{\partial}{\partial x^j}\left[u^j(x^k,t,\alpha)\cdot u^i(x^k,t,\alpha)\right] \\
&\frac{\partial}{\partial x^j}\,u^j(x^k,t,\alpha) = 0
\end{aligned}
\tag{6.2.1}
$$

Suppressing parameters yields the compact form:

$$
\begin{aligned}
&\left(\partial_t - \nu\nabla^2\right)u^i - \partial_{x^i}p = -\partial_{x^j}\left[u^j u^i\right] \\
&\partial_{x^j}u^j = 0
\end{aligned}
\tag{6.2.2}
$$

Insert the **HPF** expansion (6.1.3) into the equations of motion (6.2.1) and equate like order functionals[16] on both sides of the equation to obtain[17]:

$$\begin{aligned}
&\left(\partial_t - \nu\nabla^2\right) U_n^{i\beta_{\$n}}(x_{\$n}^k, t) - \frac{\partial}{\partial x^i} P_n^{\beta_{\$n}}(x_{\$n}^k, t) \\
&\qquad = -\frac{\partial}{\partial x^j} \sum_{p=1}^{p=n-1} \left[\!\left[U_p^{j\beta_{\$p}}(x_{\$p}^k, t) \cdot U_{n-p}^{i\beta_{(p+1)\$(n-p)}}(x_{(p+1)\$(n-p)}^k, t) \right]\!\right]_x \\
&\frac{\partial}{\partial x^j} U_n^{j\beta_{\$n}}(x_{\$n}^k, t) = 0
\end{aligned} \quad (6.2.3)$$

The $\mathbf{U_n^{i\beta_{\$n}}(x_{\$n}^k, t)}$ are continuous and square integrable over their domains and thus have a proper Fourier Transform, viz. $\mathbf{U_n^{i\beta_{\$n}}(\kappa_{\$n}^k, t)}$. The transformed equations corresponding to (6.2.3) are then[18]:

$$\begin{aligned}
&\left(\partial_t + \nu\left(\kappa_{\Sigma n}^m \kappa_{\Sigma n}^m\right)\right) U_n^{i\beta_{\$n}}(\kappa_{\$n}^k, t) - j\kappa_{\Sigma n}^i \cdot P_n^{\beta_{\$n}}(\kappa_{\$n}^k, t) \\
&\qquad = -\kappa_{\Sigma n}^j \cdot \sum_{p=1}^{p=n-1} \left[\!\left[U_p^{j\beta_{\$p}}(\kappa_{\$p}^k, t) \cdot U_{n-p}^{i\beta_{(p+1)\$(n-p)}}(\kappa_{(p+1)\$(n-p)}^k, t) \right]\!\right]_x \\
&\kappa_{\Sigma n}^j \cdot U_n^{j\beta_{\$n}}(\kappa_{\$n}^k, t) = 0 \\
&\kappa_{\Sigma n}^j = \sum_{p=1,n} \kappa_p^j
\end{aligned} \quad (6.2.4)$$

While these equations are complex, the first three terms simplify to:

<u>First order term</u> $\mathbf{U_1^i(\kappa^k, t)}$:

$$\begin{aligned}
&\left(\partial_t + \nu\bar{\kappa}_1^2\right) U_1^{i\beta_1}(\kappa_1^k, t) - j\kappa_1^i \cdot P_1^{\beta_1}(\kappa_1^k, t) = 0 \\
&\kappa_1^j \cdot U_1^{j\beta_1}(\kappa_1^k, t) = 0
\end{aligned} \quad (6.2.5)$$

<u>Second order term</u> $\mathbf{U_2^i(\kappa_1^k, \kappa_2^k, t)}$:

$$\begin{aligned}
&\left(\partial_t + \nu\bar{\kappa}_{12}^2\right) U_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) - j\kappa_{12}^i \cdot P_2^{\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) \\
&\qquad = -\kappa_{12}^j \cdot \left[\!\left[U_1^{j\beta_1}(\kappa_1^i, t) \cdot U_1^{i\beta_2}(\kappa_2^i, t) \right]\!\right]_\kappa \\
&\kappa_{12}^j \cdot U_2^{j\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = 0 \\
&\kappa_{12}^j \triangleq \kappa_1^j + \kappa_2^j
\end{aligned} \quad (6.2.6)$$

<u>Third order term</u> $\mathbf{U_3^i(\kappa_1^k, \kappa_2^k, \kappa_3^k, t)}$:

---

[16] Equating like order functionals is justified a priori by the fact that we are looking at very small velocities in the final stages of decay. A posteriori justification (like any Taylor series) is that it works!

[17] The special brackets $[\![\ \cdot\ ]\!]_x$ mean symmetrical real or transformed variables.

[18] Note that no convolution integrals are required!

$$\begin{aligned}&\left(\partial_t+\nu\cdot\bar{\kappa}_{123}^2\right)U_3^{i\beta_1\beta_2\beta_3}(\kappa_1^k,\kappa_2^k,\kappa_3^k,t)-j\kappa_{123}^i\cdot P_3^{\beta_1\beta_2\beta_3}(\kappa_1^k,\kappa_2^k,\kappa_3^k,t)\\&\qquad=-\kappa_{123}^j\cdot\left[\!\!\left[\begin{matrix}+U_1^{j\beta_1}(\kappa_1^k,t)\cdot U_2^{i\beta_2\beta_3}(\kappa_2^k,\kappa_3^k,t)\\+U_2^{j\beta_1\beta_2}(\kappa_1^k,\kappa_2^k,t)\cdot U_1^{i\beta_3}(\kappa_3^k,t)\end{matrix}\right]\!\!\right]_\kappa\\&\kappa_{123}^j\cdot U_3^{j\beta_1\beta_2\beta_3}(\kappa_1^k,\kappa_2^k,\kappa_3^k,t)=0\\&\kappa_{123}^j\triangleq\kappa_1^j+\kappa_2^j+\kappa_3^j\end{aligned}\qquad(6.2.7)$$

These equations can be solved in sequence subject to initial conditions. In fact, there are many types of initial conditions which can lead to a wide variety of decay laws.

6.3. Details of 1st Order Solutions {112 words}

Analysis of the first order Mass-Continuity and Navier Stokes Equations yields:

$$\begin{aligned}&j\kappa_1^i\cdot U_1^{i\beta_1}(\kappa_1^k,t)=0\\&\left(\partial_t+\nu\,\bar{\kappa}_1^{\,2}\right)U_1^{i\beta_1}(\kappa_1^k,t)+j\kappa^iP_1^{\beta_1}(\kappa_1^k,t)=0\end{aligned}\qquad\text{For each }\beta\qquad(6.3.1)$$

Divergence of the first order Navier-Stokes Equations yields:

$$\bar{\kappa}_1^{\,2}\cdot P_1^{\beta_1}(\kappa_1^k,t)=0\qquad(6.3.2)$$

The solution to these equations is:

$$\begin{aligned}&P_1^{\beta_1}(\kappa_1^k,t)=0\\&U_1^{i\beta_1}(\kappa_1^k,t)=A_1^{i\beta_1}(\kappa_1^k)\cdot e^{-\nu\bar{\kappa}_1^{\,2}t}\quad\text{with: }\kappa^j\cdot A_1^{j\beta_1}(\kappa_1^k)=0\end{aligned}\qquad(6.3.3)$$

Here $\mathbf{A_1^{i\beta}(..)}$ is a smooth, but not necessarily analytic function to be determined.

For ***Isotropic Turbulence***, it turns out that the most general form for the **A**'s is:

$$\begin{aligned}&A_1^{i\beta_1}(\kappa_1^k)=\Pi^{i\beta_1}(\kappa_1^k)\cdot A_1(\bar{\kappa}_1)+j\frac{\kappa_1^k}{\bar{\kappa}_1}\varepsilon^{i\beta_1k}B_1(\bar{\kappa}_1)\\&\Pi^{i\beta_1}(\kappa_1^k)\triangleq\left(\delta^{i\beta}-\frac{\kappa_1^i\kappa_1^{\beta_1}}{\bar{\kappa}_1^{\,2}}\right)\end{aligned}\qquad(6.3.4)$$

Taking a **β** over **β=1,2,3** makes (6.3.4) fairly simple because of the symmetries of $\mathbf{\Pi^{ij}(..)}$.

6.4. Details of 2nd Order Solutions {123 words}

The 2nd order equations have both a ***homogeneous*** and a ***particular*** solution. The homogeneous part is independent of any lower order functionals and has its own independent set of initial conditions. The particular part is a response to the drives of the product terms of lower order functionals. We are most interested in the particular part which is driven by lower order terms. We can do three things with the homogeneous solution:

1. Treat it as an independent solution with its own initial conditions.

2. Balance it against the particular solution so that the flow is entirely Gaussian at (say) **t=0**.
3. Take it as zero so we have only the primary decay law to deal with.

We choose the third alternative for this initial investigation. Then, analysis of the second order Mass-Continuity and Navier Stokes Equations for the particular solution yields:

$$\begin{aligned}&\left(\partial_t + \nu\left(\kappa_{12}^m \kappa_{12}^m\right)\right) U_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) + j\kappa_{12}^i \cdot P_2^{\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) \\ &\qquad = -j\kappa_{12}^j \cdot \underbrace{\left[\!\left[U_1^{j\beta_1}(\kappa_1^i, t)\cdot U_1^{i\beta_2}(\kappa_2^i, t)\right]\!\right]_\kappa}_{\text{Symmetrized on } \kappa\text{'s}} \\ &j\kappa_{12}^j \cdot U_2^{j\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = 0 \\ &\kappa_{12}^j \triangleq \kappa_1^j + \kappa_2^j\end{aligned} \tag{6.4.1}$$

The solution to these equations is:

$$\begin{aligned}&P_2^{\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = -\frac{\kappa_{12}^i \kappa_{12}^j}{\kappa_{12}^m \kappa_{12}^m} \cdot \left[\!\left[U_1^{p_1\beta_1}(\kappa_1^k, t)\cdot U_1^{p_2\beta_2}(\kappa_2^k, t)\right]\!\right]_\kappa \\ &U_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = B_2^{ip_1p_2}(\kappa_1^k, \kappa_2^k)\cdot U_1^{p_1\beta_1}(\kappa_1^k, t)\cdot U_1^{p_2\beta_2}(\kappa_2^k, t) \\ &\text{where,}\quad B_2^{ip_1p_2} = \frac{j\Gamma^{ip_1p_2}(\kappa_{12}^k)}{2\nu\, \kappa_1^k \kappa_2^k} \\ &\text{and,}\quad \Gamma^{ip_1p_2}(\kappa^k) = \left(\kappa^i \frac{\kappa^{p_1}\kappa^{p_2}}{\bar{\kappa}^2} - \frac{\kappa^{p_2}\delta^{ip_1} + \kappa^{p_1}\delta^{ip_2}}{2}\right)\end{aligned} \tag{6.4.2}$$

The velocities will have the form:

$$U_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = A_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k)\cdot e^{-\nu\left(\bar{\kappa}_1^{\,2} + \bar{\kappa}_2^{\,2}\right)t} \tag{6.4.3}$$

Here the **A**'s are smooth, but not necessarily analytic. After a bit of algebraic gymnastics, the resulting equations for the **U_2(..)** is:

$$\begin{aligned}&\left(\partial_t + \nu\left(\kappa_{12}^m \kappa_{12}^m\right)\right) U_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = j\Gamma^{ipq}(\kappa_{12}^k)\left[\!\left[A_1^{p\beta_1}(\kappa_1^k)\cdot A_1^{q\beta_2}(\kappa_2^k)\right]\!\right]_\kappa e^{-\nu\left(\bar{\kappa}_1^{\,2} + \bar{\kappa}_2^{\,2}\right)t} \\ &\Gamma^{ipq}(\kappa^k) = \left(\kappa^i \frac{\kappa^p \kappa^q}{\bar{\kappa}^2} - \frac{\kappa^q \delta^{ip} + \kappa^p \delta^{iq}}{2}\right)\end{aligned} \tag{6.4.4}$$

Then the solution for **A_2(..)** and **U_2(..)** is:
The particular solution for **U_2^i(..)** is therefore:

Fix this

$$\begin{aligned}&A_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k) = \Gamma^{ipq}(\kappa_{12}^k)\left[\!\left[A_1^{p\beta_1}(\kappa_1^k)\cdot A_1^{q\beta_2}(\kappa_2^k)\right]\!\right]_\kappa \\ &U_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k, t) = A_2^{i\beta_1\beta_2}(\kappa_1^k, \kappa_2^k)\cdot e^{-\nu\left(\bar{\kappa}_1^{\,2} + \bar{\kappa}_2^{\,2}\right)t} \\ &\qquad = \Gamma^{ipq}(\kappa_{12}^k)\left[\!\left[A_1^{p\beta_1}(\kappa_1^k)\cdot A_1^{q\beta_2}(\kappa_2^k)\right]\!\right]_\kappa \cdot e^{-\nu\left(\bar{\kappa}_1^{\,2} + \bar{\kappa}_2^{\,2}\right)t}\end{aligned} \tag{6.4.5}$$

This is a remarkably compact form.

With this result, the 2nd order contribution to the correlation tensor from **U_2(..)** is:

$$\begin{aligned}
\Phi_4^{ij}(\kappa_1^k,\kappa_2^k,t) &= \left[\!\!\left[\begin{array}{l} U_2^{i\beta_1\beta_2}(-\kappa_1^k,-\kappa_2^k) \\ \times U_2^{j\beta_1\beta_2}(+\kappa_1^k,+\kappa_2^k) \end{array}\right]\!\!\right]_\kappa \\
&= \left[\!\!\left[\begin{array}{l} \Gamma^{ipq}(-\kappa_{12}^k)\cdot A_1^{p\beta_1}(-\kappa_1^k)\cdot A_1^{q\beta_2}(-\kappa_2^k) \\ \times\Gamma^{jrs}(+\kappa_{12}^k)\cdot A_1^{r\beta_1}(+\kappa_1^k)\cdot A_1^{s\beta_2}(+\kappa_2^k) \end{array}\right]\!\!\right]_\kappa \cdot e^{-2\nu(\bar{\kappa}_1^2+\bar{\kappa}_2^2)t} \\
&= \left[\!\!\left[\begin{array}{l} \Gamma^{ipq}(-\kappa_{12}^k)\cdot\left(A_1^{p\beta_1}(-\kappa_1^k)\cdot A_1^{r\beta_1}(+\kappa_1^k)\cdot e^{-2\nu\bar{\kappa}_1^2 t}\right) \\ \times\Gamma^{jrs}(+\kappa_{12}^k)\cdot\left(A_1^{q\beta_2}(-\kappa_2^k)\cdot A_1^{s\beta_2}(+\kappa_2^k)\cdot e^{-2\nu\bar{\kappa}_2^2 t}\right) \end{array}\right]\!\!\right]_\kappa
\end{aligned} \qquad (6.4.6)$$

This simplifies to the compact form:

$$\begin{aligned}
&\Phi_4^{ij}(\kappa_1^k,\kappa_2^k,t) = \left[\!\!\left[\Gamma^{ipq}(-\kappa_{12}^k)\cdot\Gamma^{jrs}(+\kappa_{12}^k)\cdot\Phi_2^{pr}(-\kappa_1^k,t)\cdot\Phi_2^{qs}(+\kappa_2^k,t)\right]\!\!\right]_\kappa \\
&\quad\text{where:}\quad \Phi_2^{ij}(\kappa_1^k,t) = \left\{+\Pi^{ij}(\kappa_1^k)\cdot D_2(\bar{\kappa}_1) + je^{ijk}\cdot E_2(\bar{\kappa}_1)\right\}\cdot e^{-2\nu\bar{\kappa}_1^2 t}
\end{aligned} \qquad (6.4.7)$$

Higher order terms can be determined in like manner, and there will be other term contributing to the correlation tensor. Further results are reported in the Book, Chapters 7-11, including some axisymmetric cases and some **MFD** cases.

6.5. The Correlation Tensor {123 words}

From (6.3.4), the 1st order contribution to the correlation tensor is:

$$\begin{aligned}
\Phi_2^{ij}(\kappa_1^k,t) &= A_1^{i\beta_1}(-\kappa_1^k)\cdot A_1^{j\beta_1}(+\kappa_1^k)\cdot e^{-2\nu\bar{\kappa}_1^2 t} = \Psi_2^{ij}(\kappa_1^k)\cdot e^{-2\nu\bar{\kappa}_1^2 t} \\
&= \left[\begin{array}{l} +\Pi^{ij}(\kappa_1^k)\cdot\left(A_1(\bar{\kappa}_1)\cdot A_1(\bar{\kappa}_1)+B_1(\bar{\kappa}_1)\cdot B_1(\bar{\kappa}_1)\right) \\ -2j\varepsilon^{ijk}\dfrac{\kappa_1^k}{\bar{\kappa}}\cdot\left(A_1(\bar{\kappa}_1)\cdot B_1(\bar{\kappa}_1)\right) \end{array}\right]\cdot e^{-2\nu\bar{\kappa}_1^2 t} \\
&\triangleq \left[+\Pi^{ij}(\kappa_1^k)\cdot\Phi_{a,2}(\bar{\kappa}_1) + \frac{j\kappa_1^k}{\bar{\kappa}_1}\varepsilon^{ijk}\Phi_{b,2}(\bar{\kappa}_1)\right]\cdot e^{-2\nu\bar{\kappa}_1^2 t}
\end{aligned} \qquad (6.5.1)$$

This defines $\mathbf{\Phi_{a,2}(..)}$ and $\mathbf{\Phi_{b,2}(..)}$ to be:

$$\begin{aligned}
\Phi_{a,2}(\bar{\kappa}_1) &\triangleq \left(A_1(\bar{\kappa}_1)\cdot A_1(\bar{\kappa}_1)+B_1(\bar{\kappa}_1)\cdot B_1(\bar{\kappa}_1)\right) \\
\Phi_{b,2}(\bar{\kappa}_1) &\triangleq -2\cdot\left(A_1(\bar{\kappa}_1)\cdot B_1(\bar{\kappa}_1)\right)
\end{aligned} \qquad (6.5.2)$$

6.6. Further Results for HIT {123 words}

The 2nd order equations have both a ***homogeneous*** and a ***particular*** solution. The homogeneous part is independent of any lower order functionals and has its own independent set of initial conditions. The particular part is a response to the drives of the product terms of lower order functionals. We are most interested in the particular part which is driven by lower order terms. We can do three things with the homogeneous solution:

6.7. Conclusions and Next Steps for HIT {123 words}

The 2nd order equations have both a ***homogeneous*** and a ***particular*** solution. The homogeneous part is independent of any lower order functionals and has its own independent set of initial conditions. The particular part is a response to the drives of the product terms of lower order functionals. We are most interested in the particular part which is driven by lower order terms. We can do three things with the homogeneous solution:

7. Details of Plane Poiseuille Flow

7.1. Details of Basis {123 words}

Some of the points to cover are:

1. Choice of Wiener Process
2. Symmetries, known properties
3. Galerkin GERM process
4. Computer experiments:
    a. Range of Re
    b. 2D and 3D solutions
    c. Choice of expansion series

8. Some Closing Comments

8.1. Epilogue: What Has Been Said {137 words}

This Essay is an attempt to "de-mystify" the study of turbulence. As an Engineer, the author has tried (section 2) to state what randomness in the continuum really means. Then we survey the available inventory of mathematical methods (section 3) and choose the Wiener Machinery to describe the Random Functions inherent in turbulence. Third, (section 4) we outline the process of applying the tools to the problems. Finally, (section 5) we outline some of the results obtained by analyzing various flows.

Let us be clear from beginning to end of this essay that there may be other ways to achieve these goals, and there may be more complete descriptions of the mathematics. But our goal is simply to solve the problem of turbulence. We wish to avoid any side-tracks.

8.2. A Look to the Future {186 words}

For the moment, assume all that is said above is essentially true – then we have arrived at an analytic solution to a large class of turbulence problems.

But if there is error in the above, where can the error lie?

Are the Navier-Stokes Equations in differential form inappropriate to the cases of turbulence investigated? If so, many, many physicists will be surprised.

Are the Wiener Functionals inappropriate descriptions of the velocities in turbulence? If so, many, many mathematicians will be surprised.

Are the author's applications of the Wiener Functionals to Turbulence in error? If so, no one will be surprised, but any error should be correctable.

Are the author's analytic and numerical results in error? If so, no one will be surprised except by the fact that the answers so closely parallel experiment.

On the other hand, the history of physics tells us that new (but complicated) results are often quickly simplified by other scientists, usually with more profound results. If that history is repeated in this case, no one will be surprised and everyone will be pleased.

A. Appendix as Required

A.1. A Little History

A.2. The Wiener Machinery

B. Annotated Bibliography

B.1. Comments on the Bibliography

An extensive annotated bibliography is in preparation.

LOCAL TABLE OF CONTENTS:

1. Prologue: What We Want to Accomplish {1021 words} 1
1.1. Turbulence {145 words} 1
1.2. A Little History of Turbulence {609 words} 1
1.3. A Little History of This Research Effort {260 words} 2
2. Turbulence and Randomness {698 words} 3
2.1. The Presumed Nature of Turbulence {378 words} 3
2.2. The Presumed Nature of Randomness in Turbulence {312 words} 4
3. The Wiener Machinery In Brief {1718 words} 5
3.1. The Wiener Process: {161 words} 5
3.2. The Wiener Integral: {402 words} 5
3.3. The Wiener Machinery – Polynomial Functionals: {195 words} 6
3.4. The Wiener Machinery Extended to Multiple Parameters {281 words} 7
3.5. The Wiener Machinery Extended to Vectors {504 words} 8
3.6. The Wiener Machinery – A Tool for Turbulence {162 words} 9
4. The Wiener Machinery In Full {1718 words} 10
4.1. The Wiener Process: {161 words} 10
4.2. The Wiener Integral: {402 words} 10
4.3. The Wiener Machinery – Polynomial Functionals: {195 words} 11
4.4. The Wiener Machinery Extended to Multiple Parameters {281 words} 12
4.5. The Wiener Machinery Extended to Vectors {504 words} 13
4.6. The Wiener Machinery – A Tool for Turbulence {162 words} 14
5. Overview – Solutions for Two Specific Flows 15
5.1. Applying Wiener Functionals to Turbulence {347 words} 15
5.2. The Two Specific Cases {205 words} 16
5.3. Decaying Isotropic Turbulence - HIT {618 words} 16
5.4. Plane Poiseuille Flow {563 words} 17
6. Details of Isotropic Turbulence 18
6.1. Details of Basis {257 words} 18
6.2. Details of Solution Procedure {183 words} 20
6.3. Details of 1st Order Solutions {112 words} 22
6.4. Details of 2nd Order Solutions {123 words} 22
6.5. The Correlation Tensor {123 words} 24
6.6. Further Results for HIT {123 words} 24
6.7. Conclusions and Next Steps for HIT {123 words} 25
7. Details of Plane Poiseuille Flow 25
7.1. Details of Basis {123 words} 25
8. Some Closing Comments 25
8.1. Epilogue: What Has Been Said {137 words} 25
8.2. A Look to the Future {186 words} 25
A. Appendix as Required 27
A.1. A Little History 27
A.2. The Wiener Machinery 27

B. Annotated Bibliography 28
B.1. Comments on the Bibliography..28

(8623 words)