From Reynolds to Wiener
Is There a Solution to PFF Turbulence?
Maybe!

Abstract/Summary:

Yaglom and Lumley in "A Century of Turbulence, 2001" whimsically state:

> "We believe that, even after 100 years, turbulence studies are still in their infancy. We are naturalists, observing butterflies in the wild. We are still discovering how turbulence behaves, in many respects. We do have a crude, practical, working understanding of many turbulence phenomena, but certainly nothing approaching a comprehensive theory, and nothing that will provide predictions of an accuracy demanded by engineers."

We will show that there ***IS*** a comprehensive theory which provides a complete solution to certain turbulence problems.

We show the existence of such a theory. We do ***not*** (yet) provide the complete answer.

The cadence of this paper is VERY rapid – some references, no proofs.

1. The Base Case

1.1. The DNS Studies

We study steady state turbulence in the **PPF** configuration – fully-developed turbulence in a high aspect ratio rectangular channel with **x**-streamwise, **y**-cross channel (±L), **z**-spanwise. Constant viscosity and density, incompressible, stationary on time, stationary/homogeneous on **x** and **z**. "Reynolds Averaging" applies. Our parlance is that the flow is ***stationary*** and ***ergodic*** on **(x,z,t)**.

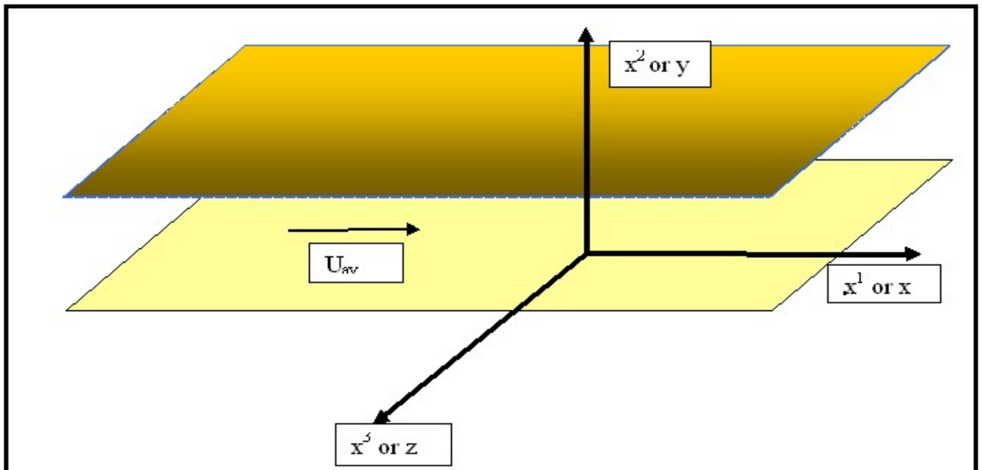

Figure 1, Channel Flow between parallel plates
"Plane Poiseuille Flow"

Over the past few years, there have been several comprehensive **DNS** studies of **PPF** over a range of Reynolds numbers[1]. These studies provide an excellent database for developing insight into the functions needed to describe **PPF** turbulence.

1.2. The Random Quantities We Want

Each **DNS** run yields a sample $\mathbf{u^i(x,y,z,t)}$ – one member of an ensemble of solutions. Following Wiener and Lumley[2], we choose to represent the ensemble of velocities by $\mathbf{u^i(x,y,z,t,\alpha)}$ where the parameter $\alpha\in[0,1]$ selects a sample from the ensemble. Thus, any average is obtained by simple integration:

$$\overline{\text{foo}}(y) = \int_0^1 \text{foo}(x,y,z,t,\alpha)\,d\alpha \qquad (1.2.1)$$

Our goal is to show that formulas exist to represent $\mathbf{u^i(x,y,z,t,\alpha)}$ with a few smooth functions and the Wiener Process $\mathbf{r(s,\alpha)}$ – aka Brownian motion.

2. Using the cdf – Cumulative Distribution Function

2.1. A Simple Scalar Velocity Example

Ultimately, we want to find an equation for $\mathbf{u^i(x,y,z,t,\alpha)}$ with $\alpha\in[0,1]$. To illustrate the process, we pick a scalar velocity at a fixed point in (**x,y,z**) space in a direction specified by the unit vector λ^i thus:

$$u(t,\alpha) = \lambda^i u^i(x,y,z,t,\alpha) \qquad (2.1.1)$$

So, $\mathbf{u(t,\alpha)}$ is stationary and ergodic on "**t**".

This random function has a **cdf^u(..)** (***cumulative distribution function for u***) defined as:

$$\mathcal{Pr}\left\{u(t,\alpha)\le\lambda\right\} = \text{cdf}^u(\lambda) \quad \text{stationary, i.e. not-dependent on "t"} \qquad (2.1.2)$$

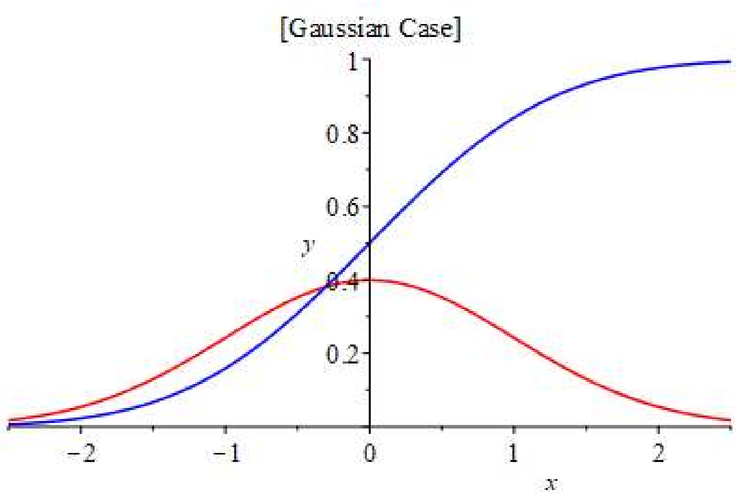

Figure 2.1. The **CDF** and **pdf** for the ***Standard Normal Gaussian Distribution***. In this case the mean is 0 and the standard deviation is 1

[1] An excellent example is the Lee-Moser (http://journals.cambridge.org/article_S0022112015002682) data for Re_tau=1995 which yields a Re_ml=43478. This case dimensions are Lx,Ly,Lz=8pi,2,3pi and point densities are nx,ny,nz=4096,768,3072. Other interesting parameters are: U_mean=1, nu=2.3e-5, u_tau=4.58794. This is a steady, strongly turbulent flow.

[2] Also, by Steinhaus, Kolmogorov, and Yaglom – inter alia.

For physical situations, **cdf^u(..)** has very "nice" properties – it is monotone strictly increasing with a **range=[0,1]** and **domain=[±∞]**. It has a smooth derivative known as the **pdf^u(..)** (***probability density function for u***), often shaped like the "bell" curve.

2.2. The Associated "quantile" Random Function

We construct the associated "***quantile***" Random Function as follows:

$$q(t,\alpha) = \mathrm{cdf}^u\left(u(t,\alpha)\right) \qquad (2.2.1)$$

What is the ***Cumulative Distribution Function*** (**cdf**) of **q**?

$$\begin{aligned}
&\mathcal{Pr}\{u(t,\alpha) \le \lambda\} = \mathrm{cdf}^u(\lambda) && \text{Standard Definition} \\
&\left\{u(t,\alpha) \le \lambda\right\} \Rightarrow \left\{\mathrm{cdf}^u(u(t,\alpha)) \le \mathrm{cdf}^u(\lambda)\right\} && \text{Apply } \mathrm{cdf}^u(\lambda) \text{ to both sides} \\
&\mathcal{Pr}\{\underbrace{\mathrm{cdf}^u(u(t,\alpha))}_{=q(t,\alpha)} \le \underbrace{\mathrm{cdf}^u(\lambda)}_{\triangleq\mu}\} = \underbrace{\mathrm{cdf}^u(\lambda)}_{\triangleq\mu} && \text{Because } \mathrm{cdf}^u(\lambda) \text{ is monotone} \\
&\mathcal{Pr}\{q(t,\alpha) \le \mu\} = \mu && \text{!!! The Uniform Distribution}
\end{aligned} \qquad (2.2.2)$$

So, **q(t,α)** has a linear-rectangular distribution, regardless of the original **u(t,α)!**

2.3. The Associated "Normal" Random Function

Now the ***Standard Normal*** **cdf^N(..)** and its inverse (quantile function) are[3]:

$$\begin{aligned}
&\mathrm{cdf}^N(\lambda) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\lambda}{\sqrt{2}}\right)\right] && \text{The "Normal" Distribution} \\
&\mathrm{icdf}^N(\mu) = \sqrt{2}\cdot \mathrm{ierf}\left(2\mu - 1\right) && \text{The "Normal" Quantile Function}
\end{aligned} \qquad (2.3.1)$$

Then, construct another ***random function*** as:

$$\phi(t,\alpha) = \mathrm{icdf}^N\left(q(t,\alpha)\right) = \mathrm{icdf}^N\left(\mathrm{cdf}^u(u(t,\alpha))\right) \qquad (2.3.2)$$

This **ϕ(t,α)** is a ***Normal Gaussian Random Function*** with **Mean=0** and **Variance=1**! Moreover, **ϕ(t,α)** is ***smooth***, ***stationary*** and ***ergodic*** on "**t**" because **u(t,α)** and **q(t,α)** are so. The inverse is:

$$u(t,\alpha) = \mathrm{icdf}^u\left(\mathrm{cdf}^N(\phi(t,\alpha))\right) \qquad (2.3.3)$$

Thus (as expected) all single point statistics – including mean, variance, skewness, kurtosis/flatness, characteristic function, etc. – are contained in **cdf^u(..)** independent of "**t**".

Summary: This demonstrates that there exist ***monotone strictly increasing functions***:

$$\begin{aligned}
&M_{u2\phi}(\lambda) \triangleq \mathrm{icdf}^N\left(\mathrm{cdf}^u(\lambda)\right) \\
&M_{\phi 2u}(\lambda) \triangleq \mathrm{icdf}^u\left(\mathrm{cdf}^N(\lambda)\right)
\end{aligned} \qquad (2.3.4)$$

[3] The Wikipedia article "Normal Distribution" is quite good.

Such that

$$\begin{aligned}\phi(t,\alpha) &= M_{u2\phi}\left(u(t,\alpha)\right) \\ u(t,\alpha) &= M_{\phi 2u}\left(\phi(t,\alpha)\right)\end{aligned} \qquad (2.3.5)$$

So, we represent $\mathbf{u(t,\alpha)}$ exactly by a Monotone Function $\mathbf{M_{\phi 2u}(..)}$ and a Normal Random Variable $\boldsymbol{\phi(t,\alpha)}$.

3. The Wiener Machinery

3.1. The Wiener Process

The "***Wiener Process***" is an idealization of "***Brownian Motion***" named for the Scottish Botanist Robert Brown. The Wiener Process $\mathbf{r(t,\alpha)}$ is continuous on $\mathbf{t}$ for any $\alpha \in \mathbf{[0,1]}$.

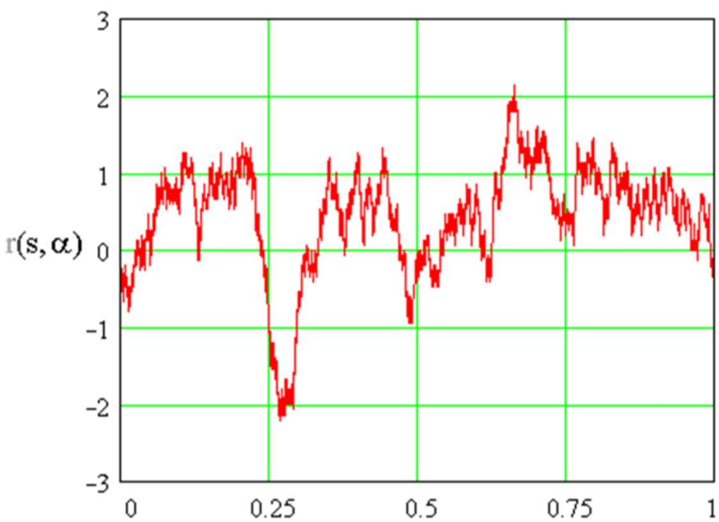

Figure 1, A Sample of the Wiener Process

Figure 3.1. A construction of the Wiener Process. Four Thousand Bernoulli Trials were integrated (summed) to form this sample function. The striking characteristics of a curve that is continuous, but not differentiable, and not bounded in variation are apparent. The fractal dimension is 3/2.

We choose a version of the function $\mathbf{r(t,\alpha)}$ which has some interesting and convenient properties. With regard to $\mathbf{t}$: for every α, $\mathbf{r(t,\alpha)}$ is: continuous on $\mathbf{t}$, unbounded in variation, and a fractal with fractal-dimension **3/2**. With regard to α: for almost all $\mathbf{t}$, $\mathbf{r(t,\alpha)}$ is: continuous on α, and for all $\mathbf{t}$ it is Square Integrable. Moreover, $\mathbf{r(t,\alpha)}$ has increments which are Gaussian distributed and are mutually independent for non-overlapping intervals.

These properties are enough to completely specify the Wiener Process $\mathbf{r(t,\alpha)}$. Most important, this simple process has enough power to be the basis of a complete representation of any Stochastic Process.

3.2. The Wiener Stochastic Convolution Integral

This ***Wiener Stochastic Convolution Integral*** is:

$$\phi(t,\alpha) = \int \Phi(t-s)\,dr(s,\alpha) \qquad \text{Usually taken over infinite limits} \qquad (3.2.1)$$

Here **$\phi(t,\alpha)$** is a ***Gaussian Process***, defined as the convolution of an ***unknown*** deterministic function **$\Phi(s)$**[4], and the ***known*** Wiener Process **$r(s,\alpha)$**[5].

It is well established[6] that every Stationary, Gaussian random function with finite mean and variance is ***completely*** specified by its mean and auto-correlation function, and any autocorrelation function can be generated with the appropriate kernel to the Wiener Integral.

Therefore, the ***Normal Gaussian Random Function*** **$\phi(t,\alpha)$** associated with the physical velocity **$u(t,\alpha)$** by the equations (2.3.5):

$$\begin{aligned}\phi(t,\alpha) &= M_{u2\phi}\left(u(t,\alpha)\right)\\ u(t,\alpha) &= M_{\phi 2u}\left(\phi(t,\alpha)\right)\end{aligned} \qquad (3.2.2)$$

Is fully represented by:

$$\phi(t,\alpha) = \int \Phi(t-s)\,dr(s,\alpha) \qquad (3.2.3)$$

With a kernel completely defined by the autocorrelation function of **$\phi(t,\alpha)$**.

<u>Conclusion</u>: Any physical velocity **$u(t,\alpha)$** which is stationary and ergodic on "**t**" and has bounded mean and variance is completely defined by two smooth functions and the Wiener Process as follows:

$$\begin{aligned} u(t,\alpha) = M_{\phi 2u}\left(\phi(t,\alpha)\right) \quad \text{where:} \quad & M_{\phi 2u}(..) \triangleq icdf^{u}\left(cdf^{N}(..)\right)\\ \text{and:} \quad & \phi(t,\alpha) = \int \Phi(t-s)\,dr(s,\alpha)\end{aligned} \qquad (3.2.4)$$

Importantly, these two functions – **$M_{\phi 2u}(..)$** and **$\Phi(..)$** – can be measured experimentally and/or by **DNS** results.

3.3. Extension to Two or More Stationary Parameters

Recall the ***Wiener Stochastic Convolution Integral*** for one stationary parameter – "**t**":

$$\phi(t,\alpha) = \int \Phi(t-s)\,dr(s,\alpha) \qquad (3.3.1)$$

We now extend this integral to ***two*** stationary parameters – "**x**" and "**t**" – as:

$$\phi(x,t,\alpha) = \int \Phi(x-x_1, t-t_1)\,dr(x_1,t_1,\alpha) \quad \text{Taken over the whole } \{x,t\} \text{ surface} \qquad (3.3.2)$$

This integral is a ***surface*** integral, and **$dr(x,t,\alpha)$** is a surface element whose variance is **$d\sigma(x,t)$** – the ***area*** of the surface element. This heuristic description can be made fully rigorous.[7]

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[4] Note that **$\Phi(s)$** is square-integrable and has a proper Fourier Transform - important for later work.

[5] Said in a jocular but informative way, we represent an ***extraordinary*** function **$f(t,\alpha)$** (perhaps a velocity) as the convolution of an unknown but ***ordinary*** function **$\Phi(..)$** with a known but ***bizarre*** function **$r(s,\alpha)$**! The Stieltjes form integral does indeed exist because **$\Phi(..)$** is locally bounded in variation and **$r(s,\alpha)$** is continuous. The literature has a persistent and annoying error in claiming that the integral does not formally exist as a Stieltjes Integral. Yaglom noted that this is an example of the classic "separation of variables" technique.

[6] See Yaglom "Stationary Random Functions", especially section 1.3.

[7] See Wiener and Poduska ScD Thesis '62. Also see RandomTheoryOfTurbulence

Of particular importance, note that **$\phi(x,t,\alpha)$** is:

1. Stationary and Homogeneous on "**x**" and "**t**" independently. So, **ALL** statistical properties of **$\phi(x,t,\alpha)$** are independent of "**x**" and/or "**t**".
2. Ergodic on "**x**" and "**t**" independently. So, Ensemble (α) Averages, Space (**x**) averages, and Time (**t**) averages are all equal.

Extension to three (or more) stationary parameter is straightforward – to be revisited below.

4. Steady 2D Plane Poiseuille Flow

4.1. Basic Random Quantities

2D flows are rare in the physical world – two examples are large scale atmospheric flows and soap-films. Nevertheless, we expect the study of **2D PPF** to yield valuable insights into the full **3D** flow. So, we examine this simplified case first before we come to the full **3D PPF** flow.

Steady, fully developed, **2D PPF** is stationary and ergodic on "**x**" and "**t**" independently. There are two velocities to consider – **$u^x(x,y,t,\alpha)$** and **$u^y(x,y,t,\alpha)$**. Conservation of Mass requires

$$\begin{aligned} &\partial_x u^x(..) + \partial_y u^y(..) = 0 && \text{Conservation of Mass} \\ &u^y(..) = -\int_{-1}^{y} \partial_x u^x(..)\,dy && \text{Satisfies } u^y(..) \text{ Boundary Conditions} \end{aligned} \quad (4.1.1)$$

So, there is ***one*** independent random function – **$u^x(x,y,t,\alpha)$** – from which **$u^y(x,y,t,\alpha)$** is readily derived[8].

4.2. The Associated quantile Function

The **$cdf^x(\lambda,y)$** of **$u^x(x,y,t,\alpha)$** is independent of "**x**" and "**t**" but is dependent on "**y**", hence:

$$\begin{aligned} &cdf^x(\lambda,y) && \text{cdf of } u^x(x,y,t,\alpha) \\ &q^x(x,y,t,\alpha) = cdf^x\left(u^x(x,y,t,\alpha),\ y\right) && \text{Associated Quantile Function} \\ &\phi^x(x,y,t,\alpha) = icdf^N\left(q^x(x,y,t,\alpha)\right) && \text{Associated Normal Function} \end{aligned} \quad (4.2.1)$$

Restated: **$\phi^x(x,y,t,\alpha)$** is a ***Normal*** Gaussian function directly derived from **$u^x(x,y,t,\alpha)$** by:

$$\begin{aligned} &\phi^x(x,y,t,\alpha) = M^x_{u2\phi}\left(u^x(x,y,t,\alpha),\ y\right) \\ &M^x_{u2\phi}(\lambda,y) = icdf^N\left(cdf^x(\lambda,y)\right) \end{aligned} \quad (4.2.2)$$

The $M^x_{u2\phi}(\lambda,y)$ is continuous on both λ and **y**, and strictly-increasing on λ Thus it has a proper λ-inverse namely $M^x_{\phi2u}(\lambda,y)$.

[8] Alternately – perhaps better – we could define one stream function **$s(x,y,t,\alpha)$** with a double order zero at **$y=\pm L$.** Then set **$u^x=+\partial_y s$** and **$u^y=-\partial_x s$** which satisfies the Conservation of Mass equation.

4.3. The Associated Normal Function

Any stationary Gaussian Random Function is completely defined by its mean and auto-correlation function[9]. The associated $\phi^x(x,y,t,\alpha)$ is ***Standard Normal*** – i.e. $\mu=0$ and $\sigma=1$ – but with an unspecified auto-correlation function. The **2D** Wiener Stochastic Convolution Integral applied to this case is:

$$\phi^x(x,y,t,\alpha) = \int \Phi^x(x-x_1,y,t-t_1)\,dr(x_1,t_1,\alpha) \qquad (4.3.1)$$

The auto-correlation function is:

$$B^x(\xi,y,\tau) = \int \Phi^x(x+\xi,y,t+\tau)\,\Phi^x(x,y,t)\,dx\,dt \qquad (4.3.2)$$

Given any $\mathbf{B^x(\xi,y,\tau)}$, a corresponding $\mathbf{\Phi^x(x,y,t)}$ can be readily defined[10].

The $\phi^x(x,y,t,\alpha)$ thus obtained is statistically identical to any other **2D** Standard Normal Random function with the same autocorrelation function $\phi^x(x,y,t,\alpha)$. Thus we can derive the random variable $\mathbf{u^x(x,y,t,\alpha)}$ from the Wiener integral by the following steps:

$$\begin{aligned} &\phi^x(x,y,t,\alpha) = \int \Phi^x(x-x_1,y,t-t_1)\,dr(x_1,t_1,\alpha) && \text{Gaussian Process} \\ &u^x(x,y,t,\alpha) = M^x_{\phi 2u}\left(\phi^x(x,y,t,\alpha)\right) && \text{Physical Velocity} \end{aligned} \qquad (4.3.3)$$

So, we claim this: With two smooth functions and an integral with a **2D** Wiener process, we can compute any statistical property of **2D PPF** flow[11]. We can also generate a "**DNS**" run with a computer-generated Wiener Process and visualize the result.

5. A Quick Look at 3D Plane Poiseuille Flow

5.1. Basic 3D Random Quantities

Incompressible **3D PPF** is completely characterized by the two velocities (say) $\mathbf{u^x(x,y,z,t,\alpha)}$ and $\mathbf{u^z(x,y,z,t,\alpha)}$. From these, the third velocity is obtained by[12]:

$$\begin{aligned} &\partial_x u^x(..) + \partial_y u^y(..) + \partial_z u^z(..) = 0 \\ &u^y(..) = \int_{-1}^{y} \left(-\partial_x u^x(..) - \partial_z u^z(..)\right) dy \end{aligned} \qquad (5.1.1)$$

We say that **3D PPF** has two degrees of freedom or **2DOF**.

If the flow were compressible, there would be **3DOF**. If the flow were incompressible **MHD** assuming Bullard's equation, there would be **4DOF**.

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[9] See Yaglom "Stationary Random Functions", especially section 1.3.

[10] Essentially, the Fourier Transform of $\mathbf{B^x(..)}$ is the Fourier Transform of $\mathbf{\Phi^x(+)*\Phi^x(-)}$.

[11] In other writings, we call this process the "***Gaussian Transform***" or ***GaussXF*** or more simply **GXF**.

[12] Alternately – perhaps better – we could define a stream function $\mathbf{s^j(x,y,z,t,\alpha)}$ with a double order zero at $\mathbf{y=\pm L}$. Then set **u=curl(s)** which satisfies the Conservation of Mass equation. We are free to set one component of $\mathbf{s^j}$ (say $\mathbf{s^2}$) to zero. In other related works, we refer to this as the **MS0S** model.

5.2. The Associated 3D quantile and Normal Functions

The **Associated Quantile Functions** are:

$$q^x(x,y,z,t,\alpha) = cdf^x\left(u^x(x,y,z,t,\alpha),\ y\right)$$
$$q^z(x,y,z,t,\alpha) = cdf^z\left(u^z(x,y,z,t,\alpha),\ y\right) \qquad (5.2.1)$$

The **Associated Normal Functions** are:

$$\phi^x(x,y,z,t,\alpha) = M^x_{u2\phi}\left(u^x(x,y,z,t,\alpha),\ y\right) \qquad M^x_{u2\phi}(\lambda,y) = icdf^N\left(cdf^x(\lambda,y)\right)$$
$$\phi^z(x,y,z,t,\alpha) = M^z_{u2\phi}\left(u^z(x,y,z,t,\alpha),\ y\right) \qquad M^z_{u2\phi}(\lambda,y) = icdf^N\left(cdf^z(\lambda,y)\right) \qquad (5.2.2)$$

These functions – $\phi^x(x,y,z,t,\alpha)$ and $\phi^z(x,y,z,t,\alpha)$ – are ***Standard Normal*** random functions, stationary and ergodic on **(x,z,t)**, and parametrically dependent on **"y"**. The appropriate Wiener Integral is:

$$\phi^i(x,t,z,y,\alpha) = \int \Phi^{i\beta}(x-x_1, t-t_1, z-z_1, y)\, dr^\beta(x_1,t_1,z_1,\alpha) \quad \text{For } i=\{x,z\},\ \beta=\{x,z\} \qquad (5.2.3)$$

Also, $\Phi^{i\beta}(..)$ can be determined from the vector autocorrelation function $B^{ij}(..)$. So, the **3D PPF** velocities are determined by 4 smooth L_2 correlation functions having 3 parameters, together with 2 smooth strictly-increasing functions $M^i_{\phi 2u}(\lambda,y)$ having 2 parameters. Schematically, the construction is:

$$\Phi^{i\beta}(x,y,z,t) \xrightarrow[\int \Phi^{i\beta} dr^\beta(..,\alpha)]{\text{Wiener Integral}} \phi^i(x,y,z,t,\alpha)$$
$$\xrightarrow[\substack{\Psi(\lambda)\\ \text{Normal cdf}(\lambda)}]{} q^i(x,y,z,t,\alpha) \qquad (5.2.4)$$
$$\xrightarrow[\text{inverse } cdf^i(\lambda,y)]{} u^i(x,y,z,t,\alpha)$$

These 6 functions are deterministic – ***Not*** Random. So, they can be determined from the equations of motion analytically if possible; approximately (say) by a Galerkin process; or numerically if necessary. Moreover, they can be measured from experiment or **DNS** runs.

5.3. Comments on 3D PPF Solution

Comparing 3D PPF to 2D PPF: The two sections above show that **3D PPF** is much more complex ***in detail*** – 4 correlation function vs. 1, and 2 **cdf** functions vs. 1. But, **3D PPF** involves no greater ***conceptual*** complexity. This supports – if not justifies – the initial study of the **2D** cases for guidance in the study of the full **3D** cases – M. Lesieur and U. Frisch seem to agree.

6. Summary and Conclusions

6.1. Basic Results

We have demonstrated that there is a solution to many stationary and ergodic turbulent flows which consists of:

1. A few very regular real functions
2. Wiener's Stochastic Convolution Integral

This solution allows us to calculate and visualize any specific instance of a flow – as does a **DNS** run.

For example, for **2D PPF**, the method used is this schematically:

$$\begin{aligned} &u = M(\phi) \quad \text{Physical Velocity from Normal Process} \\ &\phi = \int \Phi \, dr \quad \text{Normal Gaussian Process from Wiener Integral} \end{aligned} \tag{6.1.1}$$

Or more formally and precisely:

$$\begin{aligned} &u^x(x,y,t,\alpha) = M^x_{\phi 2u}\left(\phi^x(x,y,t,\alpha),\ y\right) \qquad u^x \text{ from } \phi^x \\ &\phi^x(x,y,t,\alpha) = \int \Phi^x(x - x_1, y, t - t_1)\, dr(x_1, t_1, \alpha) \qquad \phi^x \text{ from Wiener Integral} \end{aligned} \tag{6.1.2}$$

There are two very ordinary functions involved:

1. $M^x_{\phi 2u}(\lambda, y)$ very smooth on both parameters and strictly increasing on λ
2. $\Phi^x(x,y,t)$ very smooth and square integrable on all parameters

For **3D PPF**, a similar analysis leads to two **M**'s and four **Φ**'s with similar smooth properties.

6.2. Next Steps

This analysis leads to (at least) three fruitful avenues to explore:

1. Use **DNS** results to numerically determine the **M**'s and **Φ**'s for several **Reynolds** Numbers
2. Approximate **M**'s and **Φ**'s with parameterized functions and Optimize by a **Galerkin** Process
3. Attempt to solve the equations of motion directly for the **M**'s and **Φ**'s

For 1: We intend to develop **2D PPF** runs with modest accuracy over a range of Reynolds Numbers. This will provide a basis for seeing the dependence of the **M**'s and **Φ**'s on **Re** and **y**.

For 2: Guided by the results of **"1"** above, we intend to use standard models for both **M**'s and **Φ**'s – e.g. ***Pearson (parameterized) Distributions*** for **M**'s and various rapidly decreasing formulations of the **Φ**'s.

For 3: This will be an extremely difficult task. We do have guidance from **"1"** and **"2"** above, as well as a host of symmetry and boundary conditions, but the task is daunting even with Symbolic Math packages, e.g. Maple. The prospects for Isotropic turbulence are somewhat brighter because there is only one M function and the **Φ**'s must have an Isotropic structure.

More to come!

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